Arithmetic inner product formula for unitary groups

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Abstract

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We study central derivatives of L-functions of cuspidal automorphic representations for unitary groups of even variables defined over a totally real number field, and their relation with the canonical height of special cycles on Shimura varieties attached to unitary groups of the same size. We formulate a precise conjecture about an arithmetic analogue of the classical Rallis' inner product formula, which we call arithmetic inner product formula, and confirm it for unitary groups of two variables. In particular, we calculate the Néron–Tate height of special points on Shimura curves attached to certain unitary groups of two variables.

For an irreducible cuspidal automorphic representation of a quasi-split unitary group, we can associate it an ϵ-factor, which is either 1 or -1, via the dichotomy phenomenon of local theta liftings. If such factor is -1, the central L-value of the representation always vanishes and the Rallis' inner product formula is not interesting. Therefore, we are motivated to consider its central derivative, and propose the arithmetic inner product formula. In the course of such formulation, we prove a modularity theorem of the generating series on the level of Chow groups. We also show the cohomological triviality of the arithmetic theta lifting, which is a necessary step to consider the canonical height. As evidence, we also prove an arithmetic local Siegel–Weil formula at archimedean places for unitary groups of arbitrary sizes, which contributes as a part of the local comparison of the conjectural arithmetic inner product formula.

Contents

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Chapter 1

Introduction

A central question in Number Theory is to solve Diophantine equations, that is, to study the polynomial equations in the field of rational numbers, or more generally, in number fields. From the viewpoint of algebraic geometry, the zero locus of a set of polynomial equations in an affine or projective space defines naturally some geometric object, which is called an algebraic variety. Therefore, it is important to study geometric objects defined over number fields. Among them, there is a special class of algebraic varieties, called Shimura varieties, for which one can systematically construct a large supply of points (*i.e.* solutions), or more generally, cycles (*i.e.* families of solutions).

In this article, we study Shimura varieties associated to certain unitary groups and their special cycles. Moreover, we relate the arithmetic of such cycles to L-functions of automorphic representations, which are analytic objects. The method we use to set up such a relation is an arithmetic analogue of the theta lifting in the classical theory of automorphic representation. This is first observed by S. Kudla [Kud1997, Kud2002, Kud2003] and later developed by S. Kudla, M. Rapport and T. Yang [KRY2006]. We formulate a precise conjecture about an arithmetic analogue of the classical Rallis' inner product formula, which we call arithmetic inner product formula, and confirm it for unitary groups of two variables. In particular, we calculate the Néron–Tate height of special points on Shimura curves attached to certain unitary groups of two variables.

Rallis' inner product formula

Let us briefly review the Rallis' inner product formula in the classical theory of theta lifting, which first appeared in [Ral1984]. The original formula is to calculate the Petersson inner product of two automorphic forms on an orthogonal group that are lifted from a symplectic group through theta lifting. It turns out, using the Siegel–Weil formula, that the inner product is related to a diagonal integral on the doubling symplectic group of the original automorphic forms with certain Eisenstein series. This doubling method was later generalized to other cases by S. Gelbart, I. Piatetski-Shapiro and S. Rallis [GPSR1987]. This diagonal integral is in fact Eulerian. In other words, it decomposes into so-called local zeta integrals. These local zeta integrals are directly related to the L-factors of the corresponding representations. In fact, they prove in many cases that when everything is unramified, the local zeta integral coincides with the local Langlands L-factor, modified by some Tate L-factors. Later, J. Li [Li1992] extended such results to unitary groups.

In the introduction, we only look at a special case of Rallis' inner product formula, which is parallel to the arithmetic theory developed later. Let F be a totally real field, and E/F a totally imaginary quadratic extension. For an integer $n \geq 1$, let $H' = \mathrm{U}(n,n)_F$ be the unique quasi-split unitary group of a skew-hermitian space over E (with respect to the Galois involution of E/F) of rank $2n$. Let H be the unitary group of a hermitian space V over E of rank $2n$. Both H' and H are reductive groups over F. We have a Weil representation [1] ω of $H'(\mathbb{A}_F) \times H(\mathbb{A}_F)$, realizing on $\mathcal{S}(V(\mathbb{A}_E)^n)$: the space of Schwartz functions on $V(\mathbb{A}_E)^n$. For such an element $\phi \in \mathcal{S}(V(\mathbb{A}_E)^n)$, we have the following theta series

$$\theta(g,h;\phi) = \sum_{x \in V^n(E)} \left(\omega(g,h)\phi\right)(x),$$

which is a smooth, slowly increasing function on $H'(F)\backslash H'(\mathbb{A}_F) \times H(F)\backslash H(\mathbb{A}_F)$. Let $\pi \subset \mathcal{A}_0(H')$ be an irreducible representation of $H'(\mathbb{A}_F)$ contained in the space of cusp forms of H'. For every $f \in \pi$ and $\phi \in \mathcal{S}(V(\mathbb{A}_E)^n)$, we define the theta lifting to be

$$\theta_\phi^f(h) = \int_{H'(F)\backslash H'(\mathbb{A}_F)} \theta(g,h;\phi) f(g) \mathrm{d}g.$$

Similarly, we have $\theta_{\phi^\vee}^{f^\vee}$ for $f^\vee \in \pi^\vee = \left\{\overline{f} \mid f \in \pi\right\}$ and $\phi^\vee \in \mathcal{S}(V(\mathbb{A}_E)^n)$, viewed as the underlying space of the contragredient representation $\omega^\vee$. Applying a regularized Siegel–Weil formula of A. Ichino

[1] It depends on the choice of a nontrivial additive character ψ of $F\backslash\mathbb{A}_F$, and two characters χ_α ($\alpha = 1,2$) of $E^\times\mathbb{A}_F^\times\backslash\mathbb{A}_E^\times$. For simplicity, we will assume that χ_α are both trivial.

[Ich2004, Ich2007], we obtain the following formula[2]

$$\langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H = \frac{L(\frac{1}{2}, \pi)}{2\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v \in S} Z^*(0, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee), \tag{1.1}$$

where

- $\langle -, - \rangle_H$ denotes the Petersson inner product on H (for a suitable Haar measure);
- S is a finite set of primes of F containing all archimedean places;
- $L(s,\pi) = L_S(s,\pi)L^S(s,\pi)$ is the L-function of π, whose unramified part $L^S(\pi)$ is defined from the Satake parameter and the ramified part $L_S(\pi)$ is defined in [HKS1996] as a greatest common divisor;
- $\epsilon_{E/F}$ is the quadratic character of $F^\times \backslash \mathbb{A}_F^\times$ associated to the quadratic extension E/F; and
- $Z^*(0, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$ is a (normalized) local zeta integral for a ramified place $v \in S$.

For each place v of F, we can define a factor $\epsilon(\pi_v) \in \{\pm 1\}$, which is 1 if $v \notin S$. By the theta dichotomy, $Z^*(0, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$ is not always zero if and only if $\epsilon(\pi_v) = \eta_{E_v/F_v}((-1)^n \det V_v)$. We let $\epsilon(\pi) = \prod_v \epsilon(\pi_v)$. There are two cases.

1. $\epsilon(\pi) = 1$. Then we can choose a suitable hermitian space V such that the set of theta lifting θ_ϕ^f for $\phi \in \mathcal{S}(V(\mathbb{A}_E)^n)$ and $f \in \pi$ contains a nonzero function if and only if $L(\frac{1}{2}, \pi) \neq 0$.
2. $\epsilon(\pi) = -1$. Then whatever V we choose, all theta lifting θ_ϕ^f is trivial. In this case, Rallis' inner product formula is not interesting. In fact, as we will see in Theorem 2.3.9, $L(\frac{1}{2}, \pi) = 0$ for such π.

Therefore, it is natural in case (2) to ask the information about $L'(\frac{1}{2}, \pi)$. Parallel to the classical theory where the central value of the L-function relates to the lifting on the level of functions, we propose an arithmetic theory where the central *derivative* of the L-function relates to the lifting on the level of cycles. Such formulation will be elaborated in the next subsection.

Arithmetic inner product formula

We now consider the second case, that is, π has the factor $\epsilon(\pi) = -1$. Therefore, $L(\frac{1}{2}, \pi) = 0$. We also assume that the archimedean component π_∞ of π is a discrete series representation of certain type.

[2] Here, we assume that V is anisotropic for simplicity to avoid the regularization process.

From such π, we can construct, instead of the hermitian space V over E, a hermitian space (or rather a hermitian module) $\mathbb{V}$ over $\mathbb{A}_E$ of rank $2n$. Such $\mathbb{V}$ is *incoherent* in the sense that there does *not* exist a hermitian space V over E making $\mathbb{V} \cong V \otimes_E \mathbb{A}_E$, which is parallel to the fact that $\epsilon(\pi) = -1$. Moreover, by our assumption on π_∞, $\mathbb{V}$ is totally positive definite. Let $\mathbb{H} = \mathrm{U}(\mathbb{V})$ be the group of isometry of $\mathbb{V}$, which is a reductive group over $\mathbb{A}_F$. By the theory of Shimura variety, we attach to $\mathbb{H}$ (a projective system of) Shimura varieties $(\mathrm{Sh}(\mathbb{H})_K)_K$ for open compact subgroups $K \subseteq \mathbb{H}(\mathbb{A}_{F,\mathrm{fin}})$. They are smooth quasi-projective varieties over $\operatorname{Spec} E$. Let $\mathcal{S}(\mathbb{V}(\mathbb{A}_E)^n)^{\mathrm{U}_\infty} \subset \mathcal{S}(\mathbb{V}(\mathbb{A}_E)^n)$ be the subspace of those Schwartz functions whose archimedean components are essentially the Gaussian. Following S. Kudla, for $\phi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_E)^n)^{\mathrm{U}_\infty}$, we define the generating series $Z_\phi(g)$, which is a "function" on $H'(\mathbb{A}_F)$ whose values are formal series in $\mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))_\mathbb{C}$, the injective limit of groups of Chow cycles (with coefficients in $\mathbb{C}$) of codimension n on $\mathrm{Sh}(\mathbb{H})_K$ for all K. The series $Z_\phi(g)$ should be viewed as the arithmetic analogue of the classical theta series $\theta(g, \bullet; \phi)$, where the later is a "function" on $H'(\mathbb{A}_F)$ whose values are automorphic forms of H. Parallel to the automorphy property of $\theta(g, \bullet; \phi)$ (as a "function" on $H'(\mathbb{A}_F)$), we have the following result.

Theorem (Modularity of the generating series, Theorem 3.1.6)**.** *Let l be a linear functional on $\mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))_\mathbb{C}$. Then*

1. *If $l(Z_\phi)(g)$ is absolutely convergent, it is an automorphic form of H'.*
2. *If $n = 1$, $l(Z_\phi)(g)$ is absolutely convergent for every l.*

In fact, the above theorem holds for all codimensions, not just n. There is also a version in the case of symplectic-orthogonal pairs, which is proved by X. Yuan, S.-W. Zhang and W. Zhang [YZZ2009]. The proof for both cases use the induction process on the codimension, which originally comes from the idea of W. Zhang [Zha2009]. Moreover, the proof for the case where the generating series has codimension 1 reduces to the result in [YZZ2009].

Recall that in the case of classical theta lifting, we construct θ_ϕ^f simply by taking the inner product of f and the theta series. In view of the above theorem, we have the following parallel definition. We define the *arithmetic theta lifting* to be

$$\Theta_\phi^f = \int_{H'(F)\backslash H'(\mathbb{A}_F)} f(g) Z_\phi(g) \mathrm{d}g,$$

for $f \in \pi$. Rigorously speaking, the above integration is formal and we should justify such expression by applying a linear functional l as in the above theorem. Nevertheless, the (Betti) cohomology class $\mathrm{cl}(\Theta_\phi^f)$ of Θ_ϕ^f is always well-defined. To find an arithmetic analogue of the Petersson inner product, we invoke the conjectural Beilinson–Bloch height pairing [Beĭ1987, Blo1984], which is, in this particular case, a hermitian pairing

$$\langle -,-\rangle_{\mathrm{BB}} : \mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))^0_{\mathbb{C}} \times \mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))^0_{\mathbb{C}} \to \mathbb{C}.$$

Here, $\mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))^0_{\mathbb{C}} \subset \mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))_{\mathbb{C}}$ is the kernel of the cycle class map cl. Such pairing is conjectured to be positive definite. Even modulo the conjectural construction, there are still two remaining issues. First, we need $\mathrm{Sh}(\mathbb{H})_K$ to be *proper*. Second, we would like to have $\Theta^f_\phi \in \mathrm{CH}^n(\mathrm{Sh}(\mathbb{H}))^0_{\mathbb{C}}$. The first issue will be discussed in 3.2.1, where we propose some constructions as well as some conjectures. To simplify the discussion in the introduction, we assume that $\mathrm{Sh}(\mathbb{H})_K$ is already proper, which is the case when, for example, $F \neq \mathbb{Q}$. Then we have the following result concerning the second issue.

Proposition (Proposition 3.3.4)**.** *Assume that* $\mathrm{Sh}(\mathbb{H})_K$ *is proper for all* K*. Then the cohomology class* $\mathrm{cl}(\Theta^f_\phi)$ *is trivial.*

In view of the above result, we formulate the following conjecture for the *arithmetic inner product formula.*

Conjecture (Arithmetic inner product formula, Conjecture 3.3.6)**.** *Let* π *be an irreducible cuspidal automorphic representation of* $H'(\mathbb{A}_F)$ *as above. In particular,* $\epsilon(\pi) = -1$*. Then for every* $f \in \pi$*,* $f^\vee \in \pi^\vee$ *and every* $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V}(\mathbb{A}_E)^n)^{\mathrm{U}_\infty}$*, we have*

$$\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{BB}} = \frac{L'(\frac{1}{2},\pi)}{\prod_{i=1}^{2n} L(i,\epsilon^i_{E/F})} \prod_{v\in S} Z^*(0, f_v, f^\vee_v, \phi_v \otimes \phi^\vee_v),$$

where S *is a finite set of primes, and the local factors* Z^* *are same to those in the Rallis inner product formula* (1.1).

The main result of the article is the following theorem, which justifies the above conjecture in the case $n = 1$. We point out that when $n = 1$, $\mathrm{Sh}(\mathbb{H})_K$ is a curve. Therefore, the Beilinson–Bloch height pairing is simply the well-known Néron–Tate height pairing, denoted by $\langle -,-\rangle_{\mathrm{NT}}$.

Theorem (Arithmetic inner product formula, Theorem 7.2.1)**.** *Let* $n = 1$*. Let* π *be an irreducible cuspidal automorphic representation of* $H'(\mathbb{A}_F)$ *as in the above conjecture. Then for every* $f \in \pi$*,* $f^\vee \in \pi^\vee$ *and every* $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V}(\mathbb{A}_E)^n)^{\mathrm{U}_\infty}$*, we have*

$$\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \frac{L'(\frac{1}{2},\pi)}{L_F(2)L(1,\epsilon_{E/F})} \prod_{v\in S} Z^*(0, f_v, f^\vee_v, \phi_v \otimes \phi^\vee_v),$$

where S is a finite set of primes, and the local factors Z^* are same to those in the Rallis inner product formula (1.1).

The first appearance of such arithmetic analogue of Rallis' inner product formula is the main result of [KRY2006]. The authors studied the case where f is a new form of $\mathrm{PGL}_2(\mathbb{Q})$ of weight 2 and square-free level. In particular, the corresponding Shimura curve where the height pairing is taken on is the one attached to a division quaternion algebra over $\mathbb{Q}$. More recently, J. Bruinier and T. Yang [BY2009] studied the case where the Shimura curve is the modular curve. They obtain a formula that is very similar to ours here, and from which they deduce certain cases of the Gross–Zagier formula. In fact, the study of derivative of L-functions was initiated by Gross and Zagier about thirty years ago in the pioneer paper [GZ1986]. The recent work of X. Yuan, S.-W. Zhang and W. Zhang [YZZa] generalizes this formula in an extremely broad and conceptual form, based on the connection with the representation theory of the so-called restriction problem. As for this work, we generalize the formulas of Kudla et al. in a uniform and explicit form, by exploring the theory of local theta correspondence, as we will see in the next subsection where we outline the idea of the proof.

Outline of the proof

The proof of the main theorem consists of six parts, which occupy the following six chapters respectively. Throughout the process, we make our argument as general as possible. In other words, we deal with the problem for general n, not just 1, once we are able to do so.

In Chapter 2, we introduce analytic kernel functions that compute the derivative of L-functions. Such kernel functions are derivatives of Siegel Eisenstein series associated to degenerate principal series on the doubling unitary group. Precisely, given a pair of Schwartz functions $\phi_\alpha \in \mathcal{S}(\mathbb{V}(\mathbb{A}_E)^n)$ ($\alpha = 1, 2$), we have a kernel function $E'(0, g, \phi_1 \otimes \phi_2)$. We show that one can choose ϕ_α carefully such that $E'(0, g, \phi_1 \otimes \phi_2)$ can be expressed as a sum of local terms indexed by (almost all) places of F that are nonsplit in E, for g in an open dense subset. Precisely, we have the following decomposition (2.23)

$$E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = \sum_{v \notin \mathsf{S}} E_v(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2),$$

where S is a finite set of finite places of F at which ramification occurs. To compute the actual derivative $L'(\frac{1}{2}, \pi)$, we only need to take the inner product of $E'(0, \bullet, \phi_1 \otimes \phi_2)$ with f and $f^\vee$. Therefore, we should study $E_v(0, g, \phi_1 \otimes \phi_2)$. Throughout this chapter, all discussions work for general n except in 2.4.3.

We introduce the Shimura varieties attached to certain unitary groups, their special cycles and generating series in Chapter 3. Moreover, we discuss the case of non-proper Shimura varieties, where we need to compactify everything we introduce above. As mentioned previously, we prove the modularity of the generating series, and the cohomological triviality of the arithmetic theta lifting. Finally, in 3.4, where we restrict ourselves to the case $n = 1$, we introduce the arithmetic kernel functions $\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$ that compute the Néron–Tate height pairing of the arithmetic theta lifting. Then modulo certain volume factors and terms that are perpendicular to f and $f^\vee$, we have the following decomposition of the arithmetic kernel function (3.15)

$$\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) = \sum_{v^\circ \in \Sigma^\circ} \mathbb{E}_{v^\circ}(g_1, g_2; \phi_1 \otimes \phi_2),$$

where Σ° is the set of places of E, and

$$\mathbb{E}_{v^\circ}(g_1, g_2; \phi_1 \otimes \phi_2) = \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{v^\circ}$$

is a local height pairing on a certain model of the Shimura curve. It is clear that to prove the main theorem, we need to compare the analytic and arithmetic kernel functions place by place. There are three cases. First, v is an archimedean place. Second, v is a finite nonsplit place that is not in S. Third, v is either finite split or in S.

The first case is treated in Chapter 4. We reduce the comparison to a local question, which can be formulated for general dimensions. We prove a formula which we call the *archimedean local arithmetic Siegel–Weil formula*. Let $m \geq 2$ be an integer. Let T be a (nondegenerate) hermitian matrix in $\mathrm{GL}_m(\mathbb{C})$ of signature $(m-1, 1)$. Let Φ^0 be the Gaussian on V^m, where V is the standard positive definite complex hermitian space of dimension m. On the one hand, we have the Whittaker integral $W_T(s, e, \Phi^0)$ that is a holomorphic function in s. It is not hard to see that $W_T(0, e, \Phi^0) = 0$. On the other hand, we define an archimedean local intersection number $H(T)_\infty$, which is the volume of the open unit ball $\mathcal{D}$ in $\mathbb{C}^{m-1}$ with respect to a star product of Green currents constructed from Kudla–Millson forms. We prove the following result.

Theorem (Archimedean local arithmetic Siegel–Weil formula, Theorem 4.3.1)**.** *Let T be a (nondegenerate) hermitian matrix in $\mathrm{GL}_m(\mathbb{C})$ of signature $(m-1, 1)$. Then we have*

$$W'_T(0, e, \Phi^0) = C_m \exp(-2\pi \operatorname{tr} T) H(T)_\infty,$$

where C_m is some nonzero constant depending only on m, not on T.

The above theorem specified to $m = 2$ will pose a close relation between $E_v(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and $\mathbb{E}_{v^\circ}(g_1, g_2; \phi_1 \otimes \phi_2)$ for an archimedean place v dividing v°. More precisely, they are related by the process of holomorphic projection, which we discuss in 7.1.

The second case is treated in Chapter 5. We prove that for a finite place $v = \mathfrak{p}$ outside S that is nonsplit in E, $E_\mathfrak{p}(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and $\mathbb{E}_{\mathfrak{p}^\circ}(g_1, g_2; \phi_1 \otimes \phi_2)$ are equal, where $\mathfrak{p}^\circ$ is the unique place of E over $\mathfrak{p}$. This comparison is again reduced to a local question as follows. For a 2-by-2 hermitian matrix T with entries in $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$, the ring of integers of $E_{\mathfrak{p}^\circ}$, such that its determinant has odd valuation, we define a number $H_\mathfrak{p}(T)$ that is some intersection multiplicity on the smooth integral model of the Shimura curve at $\mathfrak{p}^\circ$. Let V^+ be the 2-dimensional hermitian space over E with a selfdual lattice Λ^+, and Φ^{0+} the characteristic function of Λ^+. We have a $\mathfrak{p}$-adic Whittaker integral $W_T(s, e, \Phi^{0+})$, holomorphic in s. Since $\det T$ has odd valuation, $W_T(0, e, \Phi^{0+}) = 0$. We prove the following result.

Theorem (Non-archimedean local arithmetic Siegel–Weil formula, Theorem 5.2.3 and Corollary 5.3.2)**.** *Let T be a 2-by-2 hermitian matrix with entries in $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$ such that $\det T$ has odd valuation. Then we have*

$$W_T'(0, e, \Phi^{0+}) = C_\mathfrak{p} \cdot H_\mathfrak{p}(T),$$

where $C_\mathfrak{p}$ is some nonzero constant depending only on the place $\mathfrak{p}$, not on T.

In the entire chapter, we restrict ourselves to the case $n = 1$.

The third case is treated in Chapter 6. We prove that under careful choices of ϕ_α $(\alpha = 1, 2)$, there is no contribution of terms $\mathbb{E}_{v^\circ}(g_1, g_2; \phi_1 \otimes \phi_2)$ for a finite place v either split in E or in S, after taking inner product with f and $f^\vee$. This is compatible with the analytic side since the corresponding terms are all zero. We are only able to make such argument when $n = 1$. Now we come to the final stage of the proof of the main theorem, which is accomplished in Chapter 7. As we have mentioned before, we apply the holomorphic projection to the analytic kernel function to make its archimedean part coincide with the corresponding part of the arithmetic kernel function. To prove the arithmetic inner product formula in the full generality, we apply the result of multiplicity one proved in A.2. The last step is extremely crucial to make us able to avoid explicit computations at all bad places. We would like to remark that such idea originally comes from [YZZa].

Conventions and notations

Convention 1.0.1. All rings will have a unit.

Notation 1.0.2.

- We denote by $\mathbb{Z}$ the ring of (rational) integers. We let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the fields of rational, real and complex numbers, respectively.

- For a $\mathbb{Z}$-module M and a commutative ring R, we denote $M_R = M \otimes_{\mathbb{Z}} R$ the base change R-module.

- We denote by $\mathbb{A}_{\text{fin}} = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\varprojlim_N \mathbb{Z}/N\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q}$ the ring of finite adèles; $\mathbb{A}$=$\mathbb{R} \times \mathbb{A}_{\text{fin}}$ the ring of full adèles.

- For a number field K, we let $\mathbb{A}_K = \mathbb{A} \otimes_{\mathbb{Q}} K$, $\mathbb{A}_{K,\text{fin}} = \mathbb{A}_{\text{fin}} \otimes_{\mathbb{Q}} K$ and $K_\infty = \mathbb{R} \otimes_{\mathbb{Q}} K$.

- As usual, for a subset S of places, $-_S$ (resp. $-^S$) means the S-component (resp. component away from S) for the corresponding (decomposable) adèlic object; $-_\infty$ (resp. $-_{\text{fin}}$) means the infinite/archimedean (resp. finite) part.

- For a field K, we fix a separable closure K^{sep} of K and denote $\Gamma_K = \text{Gal}(K^{\text{sep}}/K)$ the Galois group of K.

- The symbol Tr and Nm stand for the trace (resp. reduced trace) and norm (resp. reduced norm), respectively, if they are applied to fields or rings of adèles (resp. simple algebras). The symbol tr stands the trace for matrices and linear transformations.

Notation 1.0.3.

- For a ring R and integers $m, n > 0$, we denote by $\text{Mat}_{m,n}(R)$ the ring of m-by-n matrices with entries in R. We also set $\text{Mat}_n(R) = \text{Mat}_{n,n}(R)$.

- We denote by $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) the identity (resp. zero) matrix of rank n.

- We denote by ${}^t g$ the transpose of a matrix g.

Definition 1.0.4.

- Let R be a commutative ring and R' an étale (commutative) algebra over R of rank 2. Let $\tau : r \mapsto r^\tau$ for $r \in R'$ be the nontrivial automorphism of R' over R. For $n \geq 0$, a *hermitian* (resp. *skew-hermitian*) *space* of rank n over R' with respect to τ is a free module V over R' of rank n equipped with a map

$$(-,-) : V \times V \to R'$$

satisfying that

1. It is R'-linear in the first variable, *i.e.* for $r \in R$ and $v, v' \in V$, $(rv, v') = r(v, v')$.

2. It is (R', τ)-linear in the second variable, *i.e.* for $r \in R$ and $v, v' \in V$, $(v, rv') = r^\tau(v, v')$.

3. It is τ-symmetric (resp. τ-antisymmetric), *i.e.* for $v, v' \in V$, $(v, v') = (v', v)^\tau$ (resp. $(v, v') = -(v', v)^\tau$).

In fact, under the assumption (3), assumptions (1) and (2) imply each other.

- A hermitian or skew-hermitian space V over R' of rank n is *nondegenerate* if their is a basis $\{v_1, \ldots, v_n\}$ of V over R' such that the matrix $((v_i, v_j))_{1 \leq i,j \leq n}$ has determinant in $R'^\times$: the group of invertible elements in R'. It is clear that this property does not depend on the choice of the basis.

- In practice, R will be a field and R'/R a (possibly split) extension of degree 2, or R the ring of adèles of a number field and R' that of a quadratic field extension. The involution τ will always be clear in the context and hence we will not say *with respect* τ in general.

- In the main part of the article, *all hermitian spaces will be of finite rank and nondegenerate.*

Notation 1.0.5. Let $r \geq 1$ be an integer, we set

$$\mathsf{w}_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix}.$$

For $0 \leq d \leq r$, we set

$$\mathsf{w}_{r,d} = \begin{pmatrix} \mathbf{1}_d & & & \\ & & & \mathbf{1}_{r-d} \\ & & \mathbf{1}_d & \\ & -\mathbf{1}_{r-d} & & \end{pmatrix}.$$

For r elements $a_1, \ldots, a_r$ in a ring, we set

$$\mathrm{diag}[a_1, \ldots, a_r] = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}$$

to be the diagonal matrix.

Notation 1.0.6. Let G be a Lie group over a local field. We denote by $\lambda_G : G \to \mathbb{C}^\times$ the modulus character of G. Precisely, for $g \in G$, we have the adjoint action Ad_g on the Lie algebra $\mathrm{Lie}\, G$. Take a nonzero Haar measure $\mathrm{d}x$ on $\mathrm{Lie}\, G$. Then $\lambda_G(g) = \mathrm{Ad}_g^* \,\mathrm{d}x/\mathrm{d}x$.

Notation 1.0.7. Let K be a field and X a scheme of finite type over $\mathrm{Spec}\, K$,

- For an integer $r \geq 0$, we denote by $\mathrm{CH}^r(X)$ the Chow cohomology group of codimension r. In practice, X will be smooth over K. Therefore, $\mathrm{CH}^r(X)$ can be canonically identified with the abelian group of Chow cycles of X of codimension r over K.
- If $r = 1$, we set $\mathrm{Pic}(X) = \mathrm{CH}^1(X)$. In other words, $\mathrm{Pic}(X)$ is the Picard group of X, which should not be understood as the Picard scheme or Picard stack of X.

Chapter 2

Doubling method and analytic kernel functions

The goal of this chapter is to introduce the analytic kernel functions. In 2.1, we review the Siegel–Weil formula and some of its generalization that are related to our problem. In 2.2, we review the theory of I. Piatetski-Shapiro and S. Rallis on the doubling method. In particular, we deduce the Rallis' inner product formula in certain cases from the doubling method and the Siegel–Weil formula. We also introduce the (normalized) local zeta integrals that will serve as the local terms in both classical and arithmetic inner product formula. In 2.3, we introduce the L-function and the formula representing it. We show that the central L-value of an automorphic representation vanishes if its global epsilon factor equals -1. Then we introduce the analytic kernel functions that compute the L-derivatives. In 2.4, we study these analytic kernel functions. We prove that for certain nice choices of test functions, the analytic kernel function can be decomposed into terms indexed by archimedean and unramified nonsplit finite places of F.

2.1 Siegel–Weil formula and generalizations

2.1.1 Degenerate principal series and Eisenstein series

Let F be a totally real number field and E a totally imaginary quadratic extension of F. We denote by τ the nontrivial element in $\mathrm{Gal}(E/F)$ and $\epsilon_{E/F} : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$ the quadratic character associated by the class field theory. Let Σ (resp. Σ_{fin}, Σ_∞) be the set of all places (resp. finite places, infinite places) of F, and Σ°, $\Sigma^\circ_{\mathrm{fin}}$, Σ°_∞ those of E. We fix a nontrivial additive character ψ of $\mathbb{A}_F/F$, that is, a continuous character $\psi : \mathbb{A}_F \to \mathbb{C}^\times$, which is trivial on F.

For a positive integer r, we denote by W_r the standard skew-hermitian space of rank $2r$ over E with respect to the involution τ, which is equipped with a skew-hermitian form $\langle -, - \rangle$ such that there is an E-basis $\{e_1, \dots, e_{2r}\}$ satisfying for $1 \le i, j \le r$,

- $\langle e_i, e_j \rangle = 0$;
- $\langle e_{r+i}, e_{r+j} \rangle = 0$;
- $\langle e_i, e_{r+j} \rangle = \delta_{ij}$.

Let $H_r = \mathrm{U}(W_r)$ be the unitary group of W_r, which is a reductive group over F. The group $H_r(F)$, in which F can be itself or its completion at some place, is generated by the parabolic subgroup $P_r(F) = N_r(F)M_r(F)$ and the element w_r. Precisely,

$$N_r(F) = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_r & b \\ & \mathbf{1}_r \end{pmatrix} \mid b \in \mathrm{Her}_r(E) \right\};$$

$$M_r(F) = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{\tau,-1} \end{pmatrix} \mid a \in \mathrm{GL}_r(E) \right\};$$

and

$$\mathsf{w}_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix}$$

as in Notation 1.0.5. Here, $\mathrm{Her}_r(E) = \left\{ b \in \mathrm{Mat}_r(E) \mid b^\tau = {}^t b \right\}$.

We fix a place $v \in \Sigma$ and suppress it from notations. Thus $F = F_v$ is a local field of characteristic zero; $E = E_v := E \otimes_F F_v$ is a quadratic extension of F which could be split; and $H_r = H_{r,v} := H_r(F_v)$ is a reductive Lie group over the local field. We define $\mathcal{K}_r$ to be a maximal compact subgroup of H_r

in the following way:

- If v is finite, then

$$\mathcal{K}_r = H_r \cap \mathrm{GL}(\mathcal{O}_E\langle e_1, \dots, e_{2n}\rangle) \subset \mathrm{GL}(W_r).$$

- If v is (real) infinite, then

$$\mathcal{K}_r = H_r \cap \mathrm{U}(2r)_{\mathbb{R}} \subset \mathrm{GL}(2r)_{\mathbb{C}} \cong \mathrm{GL}(W_r)$$

  where the isomorphism is determined by the basis $\{e_1, \dots, e_{2n}\}$. Therefore, $\mathcal{K}_r$ is isomorphic to $\mathrm{U}(r)_{\mathbb{R}} \times \mathrm{U}(r)_{\mathbb{R}}$ (2.2).

For $s \in \mathbb{C}$ and a character χ of $E^\times$, we denote by $I_r(s,\chi) = \mathrm{sInd}_{P_r}^{H_r}(\chi|\bullet|_E^{s+\frac{r}{2}})$ the *degenerate principal series* representation (*cf.* [KS1997]) of H_r, where sInd stands for the (non-normalized) smooth $\mathcal{K}_r$-finite induction. Precisely, it is realized on the space of smooth $\mathcal{K}_r$-finite functions φ_s on H_r satisfying

$$\varphi_s(n(b)m(a)g) = \chi(\det a)|\det a|_E^{s+\frac{r}{2}}\varphi_s(g)$$

for all $g \in H_r$, $m(a) \in M_r$ and $n(b) \in N_r$. A (holomorphic) section φ_s of $I_r(s,\chi)$ is called *standard* if its restriction to $\mathcal{K}_r$ is independent of s. It is called *unramified* if it takes value 1 on $\mathcal{K}_r$.

Now we view F and E as number fields. For a (continuous) character χ of $\mathbb{A}_E^\times$ that is trivial on $E^\times$, and $s \in \mathbb{C}$, we have an admissible representation $I_r(s,\chi) = \bigotimes'_{v\in\Sigma} I_r(s,\chi_v)$ of $H_r(\mathbb{A}_F)$, where the restriction in the restricted tensor product refers to the collection of unramified sections. For a standard section $\varphi_s = \otimes\varphi_{s,v} \in I_r(s,\chi)$, we define the Eisenstein series

$$E(g,\varphi_s) = \sum_{\gamma \in P_r(F)\backslash H_r(F)} \varphi_s(\gamma g). \tag{2.1}$$

The series is absolutely convergent if $\mathrm{Re}\, s > \frac{r}{2}$, and has a meromorphic continuation to the entire complex plane, which is holomorphic at $s = 0$ (*cf.* [Tan1999, Proposition 4.1]).

2.1.2 Real quasi-split unitary groups

Let $r \geq 1$ be an integer. Let $W_{\mathbb{R},r}$ be a skew-hermitian space over $\mathbb{C}$ (with respect to the quadratic extension $\mathbb{C}/\mathbb{R}$) of rank $2r$ with a basis $\{e_1, \dots, e_r; e_{r+1}, \dots, e_{2r}\}$, under which the skew-hermitian form is given by the matrix

$$\mathsf{w}_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix}.$$

Let $\mathrm{U}(r,r)_{\mathbb{R}}$ be the subgroup of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathrm{GL}(W_{\mathbb{R},r})$ preserving the skew-hermitian form, which is a reductive group over $\mathbb{R}$. We also let

$$\mathrm{U}(r)_{\mathbb{R}} = \mathrm{U}(r,0)_{\mathbb{R}} = \{g \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathrm{GL}_r(\mathbb{C}) \mid {}^t\overline{g}g = \mathbf{1}_r\} \tag{2.2}$$

be a subgroup of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathrm{GL}_r(\mathbb{C})$[1]. We have the following embedding

$$\begin{aligned} \mathrm{U}(r)_{\mathbb{R}} \times \mathrm{U}(r)_{\mathbb{R}} &\to \mathrm{U}(r,r)_{\mathbb{R}} \\ (k_1,k_2) &\mapsto [k_1,k_2] := \frac{1}{2}\begin{pmatrix} k_1+k_2 & -ik_1+ik_2 \\ ik_1-ik_2 & k_1+k_2 \end{pmatrix}. \end{aligned}$$

Moreover, the above embedding identifies $\mathrm{U}(r)_{\mathbb{R}} \times \mathrm{U}(r)_{\mathbb{R}}$ as a maximal compact subgroup $\mathcal{K}_r$ of $\mathrm{U}(r,r)_{\mathbb{R}}$.

Notation 2.1.1. We introduce the following notation.

1. We let ι_i $(i = 1,\dots,d)$ be all embeddings of F into $\mathbb{C}$, whose image is contained in $\mathbb{R}$, and ι_i°, $\iota_i^\bullet$ those of E above ι_i. We identify $E \otimes_{F,\iota_i} \mathbb{R}$ with $\mathbb{C}$ through the embedding ι_i°. In particular, we have identified $H_r \times_{F,\iota_i} \mathbb{R}$ with $\mathrm{U}(r,r)_{\mathbb{R}}$.

2. Let ι be an archimedean place of F. Let χ_ι be a character of $E_\iota^\times$, which is identified with $\mathbb{C}^\times$ via ι°, such that $\chi_\iota \mid F_{\iota,>0}^\times \cong \mathbb{R}_{>0}^\times = 1$. In particular, we can write

$$\chi_\iota(z) = \frac{z^{\mathfrak{k}^{\chi_\iota}}}{\sqrt{|z\overline{z}|}^{\mathfrak{k}^{\chi_\iota}}},$$

for a unique integer $\mathfrak{k}^{\chi_\iota}$. If χ is an automorphic character of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_F^\times$ equals $\eta_{E/F}^m$, we set $\mathfrak{k}^\chi = (\mathfrak{k}^{\chi_{\iota_1}},\dots,\mathfrak{k}^{\chi_{\iota_d}})$.

Definition 2.1.2 (Weights)**.** We define the notions of weights as follows.

1. Let π be an irreducible $(\mathrm{Lie}\,\mathrm{U}(r,r)_{\mathbb{R}},\mathcal{K}_r)$-module (or its Casselman–Wallach globalization). Let a,b be integers. We say π is *of weight* (a,b) if the minimal $\mathcal{K}_r$-type of π is the character $\det^a \boxtimes \det^b$, *i.e.* its sends $[k_1,k_2]$ to $(\det k_1)^a(\det k_2)^b$.

[1]When we work in the context of $\mathbb{C}/\mathbb{R}$, a *bar* will stand for the complex conjugation unless otherwise specified.

2. Let π be an irreducible automorphic representation of $H_r(\mathbb{A}_F)$. For two d-tuples of integers $\mathbf{a} = (a_1, \ldots, a_d), \mathbf{b} = (b_1, \ldots, b_d)$, we say $\pi_\infty = \bigotimes_{i=1}^d \pi_{\iota_i}$ is *of weight* $(\mathbf{a}, \mathbf{b})$ if for every $i = 1, \ldots, d$, π_{ι_i} is of weight (a_i, b_i) in the sense above.

3. An automorphic form f of $H_r(\mathbb{A}_F)$ is *of weight* $(\mathbf{a}, \mathbf{b})$ if $f(g[k_{1,\iota}, k_{2,\iota}]) = (\det k_{1,\iota})^{a_\iota} (\det k_{2,\iota})^{b_\iota} f(g)$ for every $\iota \in \Sigma_\infty$.

2.1.3 Weil representations and theta functions

Let us review the classification of (nondegenerate) hermitian spaces. Let $m \geq 1$ be an integer and $v \in \Sigma$ be a place of F. For a hermitian space V over E_v of rank m, we define

$$\epsilon(V) = \epsilon_{E_v/F_v}\left((-1)^{\frac{m(m-1)}{2}} \det V\right) \in \{\pm 1\}.$$

There are three cases:

- If $v \in \Sigma_{\mathrm{fin}}$ such that E is nonsplit at v, then up to isometry, there are two different hermitian spaces over E_v of dimension $m \geq 1$: $V^{\pm}$ determined by $\epsilon(V^{\pm}) = \pm 1$.

- If $v \in \Sigma_{\mathrm{fin}}$ such that E is split at v, then up to isometry, there is only one hermitian space V^+ over E_v of dimension m, and $\epsilon(V^+) = 1$.

- If $v \in \Sigma_\infty$ (which is real), then up to isometry, there are $m+1$ different hermitian spaces over E_v of dimension m: V_s with signature $(s, m-s)$ where $0 \leq s \leq m$, and $\epsilon(V_s) = (-1)^{\frac{m(m+1)}{2} - s}$.

In the global situation, up to isometry, all hermitian spaces V over E of dimension m are classified by signatures at infinite places and $\det V \in F^\times / \mathrm{Nm}\, E^\times$. In particular, V is determined by $V_v = V \otimes_F F_v$ for all $v \in \Sigma$.

More generally, we also need to consider nondegenerate hermitian spaces over $\mathbb{A}_E$ of rank m. Recall in Definition 1.0.4 that in this case, a hermitian space $\mathbb{V}$ is *nondegenerate* if there is a basis under which the matrix representing the hermitian form is invertible in $\mathrm{GL}_m(\mathbb{A}_E)$. For a place $v \in \Sigma$, we let $\mathbb{V}_v = \mathbb{V} \otimes_{\mathbb{A}_F} F_v$, $\mathbb{V}_{\mathrm{fin}} = \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_{F,\mathrm{fin}}$; and define $\Sigma(\mathbb{V}) = \{v \in \Sigma \mid \epsilon(\mathbb{V}_v) = -1\}$, which is a finite set. Finally, we let $\epsilon(\mathbb{V}) = \prod_{v \in \Sigma} \epsilon(\mathbb{V}_v)$.

Definition 2.1.3 (Coherent/incoherent hermitian spaces)**.** We say a (nondegenerate) hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ is *coherent* (resp. *incoherent*) if the cardinality of $\Sigma(\mathbb{V})$ is even (resp. odd), *i.e.* $\epsilon(\mathbb{V}) = 1$ (resp. -1).

By the Hasse principle, there is a hermitian space V over E such that $\mathbb{V} \cong V \otimes_F \mathbb{A}_F$ if and only if $\mathbb{V}$ is coherent. These two terminologies are inspired from the coherent/incoherent collections of local quadratic spaces introduced by S. Kudla in the orthogonal case in [KR1994, Kud1997].

We fix a place $v \in \Sigma$ and suppress it from notations. For a hermitian space V of dimension m with hermitian form $(-,-)$ and a positive integer r, we can construct a symplectic space $\mathbf{W} = \operatorname{Res}_{E/F} W_r \otimes_E V$ of dimension $4rm$ over F with the skew-symmetric form $\operatorname{Tr}_{E/F}\langle -,-\rangle \otimes (-,-)$, where Res stands for the Weil restriction. Let $\mathrm{Sp}(\mathbf{W})$ be the symplectic group and $\mathrm{Mp}(\mathbf{W})$ be its $\mathbb{C}^\times$-metaplectic cover fitting into the following exact sequence:

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{Mp}(\mathbf{W}) \longrightarrow \mathrm{Sp}(\mathbf{W}) \longrightarrow 1.$$

We let $H = \mathrm{U}(V)$ be the unitary group of V and $\mathcal{S}(V^r)$ the space of Schwartz functions on V^r. Given a character χ of $E^\times$ satisfying $\chi|_{F^\times} = \epsilon_{E/F}^m$, we have a splitting homomorphism

$$\tilde{\imath}_{(\chi,\mathbf{1})} : H_r \times H \to \mathrm{Mp}(\mathbf{W})$$

lifting the natural homomorphism $\imath : H_r \times H \to \mathrm{Sp}(\mathbf{W})$ (*cf.* [HKS1996, Section 1]). We thus have a *Weil representation* (with respect to ψ) $\omega_\chi = \omega_{\chi,\psi}$ of $H_r \times H$ on the space $\mathcal{S}(V^r)$. Explicitly, for $\phi \in \mathcal{S}(V^r)$ and $h \in H$,

- $(\omega_\chi(n(b))\phi)(x) = \psi(\operatorname{tr} bT(x))\phi(x)$;
- $(\omega_\chi(m(a))\phi)(x) = |\det a|_E^{\frac{m}{2}}\chi(\det a)\phi(xa)$;
- $(\omega_\chi(\mathsf{w}_r)\phi)(x) = \gamma_V \widehat{\phi}(x)$;
- $(\omega_\chi(h)\phi)(x) = \phi(h^{-1}x)$,

where

$$T(x) = \frac{1}{2}\left((x_i, x_j)\right)_{1 \le i,j \le r}$$

is the *moment matrix* of x; γ_V is the Weil constant associated to the underlying quadratic space of V (and also ψ); $\widehat{\phi}$ is the Fourier transform

$$\widehat{\phi}(x) = \int_{V^r} \phi(y)\psi\left(\frac{1}{2}\operatorname{Tr}_{E/F}(x, {}^t y)\right) \mathrm{d}y$$

using the selfdual measure $\mathrm{d}y$ on V^r with respect to ψ.

In the global situation where F is the number field, by taking the restricted tensor product over all local Weil representations, we obtain a representation of $H_r(\mathbb{A}_F) \times H(\mathbb{A}_F)$ on the space

$$\mathcal{S}(V(\mathbb{A}_E)^r) = \bigotimes_{v \in \Sigma}{}' \mathcal{S}(V_v^r).$$

For V over E, χ a character of $\mathbb{A}_E^\times / E^\times$ such that $\chi|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$ and $\phi \in \mathcal{S}(V(\mathbb{A}_E)^r)$, we define the *theta function*

$$\theta(g, h; \phi) = \sum_{x \in V^r(E)} \left(\omega_\chi(g, h)\phi\right)(x),$$

which is a smooth, slowly increasing function on $H_r(F)\backslash H_r(\mathbb{A}_F) \times H(F)\backslash H(\mathbb{A}_F)$. Consider the integral

$$I_V(g, \phi) = \int_{H(F)\backslash H(\mathbb{A}_F)} \theta(g, h; \phi)\mathrm{d}h,$$

if it is absolutely convergent. Here we normalize the measure $\mathrm{d}h$ such that $\mathrm{Vol}(H(F)\backslash H(\mathbb{A}_F)) = 1$. It is well-known that $I_V(g, \phi)$ is absolutely convergent for all ϕ if $m > 2r$ or V is anisotropic.

2.1.4 Siegel–Weil formulae

It is immediate to see that

$$\varphi_{\phi,s}(g) = \left(\omega_\chi(g)\phi\right)(0)\lambda_{P_r}(g)^{s - \frac{m-r}{2}}$$

is a standard section in $I_r(s, \chi)$ for every $\phi \in \mathcal{S}(V(\mathbb{A}_E)^r)$. Recall the following formula for the modulus character

$$\lambda_{P_r}(g) = \lambda_{P_r}(n(b)m(a)k) = |\det a|_{\mathbb{A}_E},$$

if $g = n(b)m(a)k$ under the Iwasawa decomposition with respect to the parabolic subgroup $P_r(\mathbb{A}_F)$. Therefore, we can define the Eisenstein series $E(s, g, \phi) = E(g, \varphi_{\phi,s})$ (2.1). We have the following theorem.

Theorem 2.1.4 (Siegel–Weil formula). *Let $s_0 = \frac{m-r}{2}$. Then we have*

1. *If $m > 2r$, $E(s_0, g, \phi)$ is absolutely convergent and*

$$E(s_0, g, \phi) = I_V(g, \phi).$$

2. *If $r < m \leq 2r$ and V is anisotropic, $E(s, g, \phi)$ is holomorphic at s_0 and*

$$E(s, g, \phi)|_{s=s_0} = I_V(g, \phi).$$

3. *if $m = r$ and V is anisotropic, $E(s, g, \phi)$ is holomorphic at $s_0 = 0$ and*

$$E(s, g, \phi)|_{s=0} = 2I_V(g, \phi).$$

Proof. 1. It is the classical Siegel–Weil formula.

2. It is a generalized Siegel–Weil formula proved in [Ich2007, Theorem 1.1].

3. It is a generalized Siegel–Weil formula proved in [Ich2004, Theorem 4.2].

□

In what follows, we simply write $E(s_0, g, \phi)$ for $E(s, g, \phi)|_{s=s_0}$ for simplicity if the Eisenstein series is holomorphic at s_0.

Remark 2.1.5. In Theorem 2.1.4 (3), if V is isotropic, we still have a (regularized) Siegel–Weil formula. However, since the theta integral $I_V(g, \phi)$ is not necessarily convergent, a regularization process must be applied. The inner product introduced in the next section also requires a regularization process. Since the classical inner product formula is not the purpose of this article, we will always assume that V is anisotropic for simplicity, or pretend that the regularization process has been applied for general V in the following discussion.

2.2 Doubling integrals

2.2.1 Decomposition of global period integrals

Let $m = 2n$ and $r = n$ with $n \geq 1$ and suppress n from notations except that we will use H' instead of H_n; P' instead of P_n; N' instead of N_n and $\mathcal{K}'$ instead of $\mathcal{K}_n$. Therefore, $\chi|_{\mathbb{A}_F^\times} = \mathbf{1}$ is the trivial character. Let $\pi = \bigotimes'_{v\in\Sigma} \pi_v$ be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$

contained in $\mathcal{A}_0(H')$: the space of cuspidal automorphic forms on $H'(\mathbb{A}_F)$. We will not distinguish π with its underlying space. Let $\pi^\vee$ be the contragredient representation which is realized on the space of complex conjugation of functions in π.

We denote by $(-W)$ the skew-hermitian space over E with the skew-hermitian form $-\langle -,-\rangle$. Therefore, we have a basis $\{e_1^-,\dots,e_{2n}^-\}$ satisfying for $1 \le i \le n$,

- $\langle e_i^-, e_j^-\rangle = 0$;
- $\langle e_{r+i}^-, e_{r+j}^-\rangle = 0$;
- $\langle e_i^-, e_{n+j}^-\rangle = -\delta_{ij}$.

Let $W'' = W \oplus (-W)$ be the direct sum of the two skew-hermitian spaces. There is a natural embedding

$$\imath : H' \times H' \hookrightarrow \mathrm{U}(W'') \tag{2.3}$$

which is, under the basis

- $\{e_1,\dots,e_{2n}\}$ of W and
- $\{e_1,\dots,e_n; e_1^-,\dots,e_n^-; e_{n+1},\dots,e_{2n}; -e_{n+1}^-,\dots,-e_{2n}^-\}$ of W'',

given by $\imath(g_1,g_2) = \imath_0(g_1,g_2^\vee)$ where

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},\ g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},\ g^\vee = \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix} g \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix}^{-1}$$

and

$$\imath_0(g_1,g_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

Notation 2.2.1. Let

$$W' = \mathrm{span}_E\{e_1,\dots,e_n; e_1^-,\dots,e_n^-\}$$
$$\overline{W}' = \mathrm{span}_E\{e_{n+1},\dots,e_{2n}; -e_{n+1}^-,\dots,-e_{2n}^-\},$$

be a pair of complimentary Lagrangian subspaces of W''. Let P be the parabolic subgroup of $\mathrm{U}(W'')$ stabilizing W' and N the unipotent radical of P.

For the complete polarization $W'' = W' \oplus \overline{W}'$, we can realize the Weil representation of $\mathrm{U}(W'')$, denoted by ω''_χ (with respect to ψ), on the space $\mathcal{S}(V(\mathbb{A}_E)^{2n})$, such that $\imath^*\omega''_\chi \cong \omega_{\chi,\psi} \boxtimes \chi\omega^\vee_{\chi,\psi}$, where the latter one is realized on the space $\mathcal{S}(V(\mathbb{A}_E)^n) \otimes \mathcal{S}(V(\mathbb{A}_E)^n)$. Here we realize the contragredient representation $\omega^\vee_{\chi,\psi}$ on the space $\mathcal{S}(V(\mathbb{A}_E)^n)$ through the bilinear pairing

$$\langle \phi, \phi^\vee \rangle_V = \int_{V^n(\mathbb{A}_E)} \phi(x)\phi^\vee(x)\mathrm{d}x$$

for $\phi, \phi^\vee \in \mathcal{S}(V(\mathbb{A}_E)^n)$. Then $\omega^\vee_{\chi,\psi}$ is identified with $\omega_{\chi^{-1},\psi^{-1}}$.

For $\phi \in \mathcal{S}(V(\mathbb{A}_E)^n)$ and $f \in \pi$, we define the*theta lifting*

$$\theta^f_\phi(h) = \int_{H'(F)\backslash H'(\mathbb{A}_F)} \theta(g, h; \phi) f(g)\mathrm{d}g.$$

It is a well-defined, slowly increasing function on $H(F)\backslash H(\mathbb{A}_F)$, where $\mathrm{d}g = \otimes_{v\in\Sigma}\mathrm{d}g_v$ is normalized such that $\mathcal{K}'_v$ gets volume 1 for every $v \in \Sigma$. Similarly, for $\phi^\vee \in \mathcal{S}(V(\mathbb{A}_E)^n)$ and $f^\vee \in \pi^\vee$, we have $\theta^{f^\vee}_{\phi^\vee}$. The reader should be careful that in the contragredient side, the Weil representation used to form the theta function should also be the contragredient one, *i.e.* $\omega^\vee_\chi$. We have

$$\begin{aligned}
\langle \theta^f_\phi, \theta^{f^\vee}_{\phi^\vee} \rangle_H :&= \int_{H(F)\backslash H(\mathbb{A}_F)} \theta^f_\phi(h)\theta^{f^\vee}_{\phi^\vee}(h)\mathrm{d}h \\
&= \int_{H(F)\backslash H(\mathbb{A}_F)} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} \theta(g_1, h; \phi) f(g_1)\theta(g_2, h; \phi^\vee) f^\vee(g_2)\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}h \\
&= \int_{H(F)\backslash H(\mathbb{A}_F)} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} \theta(\imath(g_1, g_2), h; \phi \otimes \phi^\vee) f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2)\mathrm{d}g_1\mathrm{d}g_2\mathrm{d}h \\
&= \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2) \int_{H(F)\backslash H(\mathbb{A}_F)} \theta(\imath(g_1, g_2), h; \phi \otimes \phi^\vee)\mathrm{d}h\mathrm{d}g_1\mathrm{d}g_2.
\end{aligned} \tag{2.4}$$

We assume that V is anisotropic. Then the inside integral in the last step is absolutely convergent. By Theorem 2.1.4 (3), we have

$$(2.4) = \frac{1}{2} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2) E(0, \imath(g_1, g_2), \phi \otimes \phi^\vee)\mathrm{d}g_1\mathrm{d}g_2.$$

We should mention that the Eisenstein series on $\mathrm{U}(W'')$ appearing above is formed with respect to the parabolic subgroup P (see Notation 2.2.1), *i.e.*

$$E(s,g,\Phi)=E(g,\varphi_{\Phi,s})=\sum_{\gamma\in P(F)\backslash \mathrm{U}(W'')(F)}\omega''_\chi(\gamma g)\Phi(0)\lambda_P(\gamma g)^s$$

for $g\in \mathrm{U}(W'')(\mathbb{A}_F)$, $\Phi\in \mathcal{S}(V(\mathbb{A}_E)^{2n})$ and $\mathrm{Re}\, s>n$. The coset $P(F)\backslash \mathrm{U}(W'')(F)$ can be canonically identified with the space of isotropic n-planes in W''. Under the right action of $H'(F)\times H'(F)$ through $\imath$, the orbit of an n-plane Z is determined by the invariant $d=\dim Z\cap W=\dim Z\cap(-W)$. Let γ_d be a representative of the corresponding double coset where $0\le d\le n$. In particular, we take

$$\gamma_0=\begin{pmatrix} & & \mathbf{1}_n & \\ & \mathbf{1}_n & & \\ -\mathbf{1}_n & \mathbf{1}_n & & \\ & & \mathbf{1}_n & \mathbf{1}_n \end{pmatrix}\text{ and }\gamma_n=\mathbf{1}_{4n}$$

(*cf.* [KR2005]). Let St_d be the stabilizer of the coset $P\gamma_d$ in $H'\times H'$. In particular $\mathrm{St}_0=\Delta(H')$ is the diagonal subgroup. Therefore, for a standard section $\varphi_s\in I_{2n}(s,\chi)$ and $\mathrm{Re}\, s>n$,

$$\begin{aligned}&\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(\imath(g_1,g_2),\varphi_s)\mathrm{d}g_1\mathrm{d}g_2\\ =&\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2}(f\otimes f^\vee\chi^{-1})(g)\sum_{\gamma\in P(F)\backslash \mathrm{U}(W'')(F)}\varphi_s(\gamma\imath(g))\mathrm{d}g\\ =&\sum_{d=0}^n\int_{\mathrm{St}_d(F)\backslash H'(\mathbb{A}_F)^2}(f\otimes f^\vee\chi^{-1})(g)\varphi_s(\gamma_d\imath(g))\mathrm{d}g. \end{aligned}\tag{2.5}$$

When $d>0$, St_d has nontrivial unipotent radical. Since f and $f^\vee$ are cuspidal, we have

$$\begin{aligned}(2.5)=&\int_{\Delta(H'(F))\backslash H'(\mathbb{A}_F)^2}(f\otimes f^\vee\chi^{-1})(g)\varphi_s(\gamma_0\imath(g))\mathrm{d}g\\ =&\int_{H'(\mathbb{A}_F)}\int_{H'(F)\backslash H'(\mathbb{A}_F)} f(g_1g_2)f^\vee(g_1)\chi^{-1}(\det g_1)\varphi_s(\gamma_0\imath(g_1g_2,g_1))\mathrm{d}g_1\mathrm{d}g_2\\ =&\int_{H'(\mathbb{A}_F)}\int_{H'(F)\backslash H'(\mathbb{A}_F)}\pi(g_2)f(g_1)f^\vee(g_1)\chi^{-1}(\det g_1)\varphi_s(p(g_1)\gamma_0\imath(g_2,1))\mathrm{d}g_1\mathrm{d}g_2\end{aligned}\tag{2.6}$$

where $p(g_1)\gamma_0=\gamma_0\imath(g_1,g_1)$, and under the Levi decomposition $p(g_1)=n(b)m(a)\in P(\mathbb{A}_F)$, we have

$\det a = \det g_1$. Therefore,

$$\begin{aligned}
(2.6) &= \int_{H'(\mathbb{A}_F)} \int_{H'(F)\backslash H'(\mathbb{A}_F)} \pi(g_2) f(g_1) f^\vee(g_1) \mathrm{d}g_1 \varphi_s(\gamma_0 \imath(g_2, 1)) \mathrm{d}g_2 \\
&= \int_{H'(\mathbb{A}_F)} \langle \pi(g) f, f^\vee \rangle \varphi_s(\gamma_0 \imath(g, 1)) \mathrm{d}g \\
&= \prod_{v \in \Sigma} \int_{H'_v} \langle \pi_v(g_v) f_v, f_v^\vee \rangle \varphi_{s,v}(\gamma_0 \imath(g_v, 1)) \mathrm{d}g_v
\end{aligned}$$

where we assume that f, $f^\vee$ and φ_s are all decomposable. In summary, we have the following proposition.

Proposition 2.2.2. *Let f, $f^\vee$ and φ_s be as above. Then for $\operatorname{Re} s > n$, the integral*

$$\begin{aligned}
&\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2) E(\imath(g_1, g_2), \varphi_s) \mathrm{d}g_1 \mathrm{d}g_2 \\
= &\prod_{v \in \Sigma} \int_{H'_v} \langle \pi_v(g_v) f_v, f_v^\vee \rangle \varphi_{s,v}(\gamma_0 \imath(g_v, 1)) \mathrm{d}g_v
\end{aligned}$$

defines an element in the space

$$\mathrm{Hom}_{H'(\mathbb{A}_F) \times H'(\mathbb{A}_F)}(I_{2n}(s, \chi), \pi^\vee \boxtimes \chi\pi) = \bigotimes_{v \in \Sigma} \mathrm{Hom}_{H'_v \times H'_v}(I_{2n}(s, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v).$$

2.2.2 Local zeta integrals

We study the local functionals obtained in 2.2.1.

Fix a finite place v of F and suppress it from notations. For $f \in \pi$, $f^\vee \in \pi^\vee$ and a holomorphic section $\varphi_s \in I_{2n}(s, \chi)$, define the *local zeta integral*

$$Z(\chi, f, f^\vee, \varphi_s) = \int_{H'} \langle \pi(g) f, f^\vee \rangle \varphi_s(\gamma_0 \imath(g, 1)) \mathrm{d}g,$$

which is absolutely convergent when $\operatorname{Re} s > 2n$. In [HKS1996, Section 6], the family of *good sections* is introduced. For every good section φ_s, the zeta integral $Z(\chi, f, f^\vee, \varphi_s)$ is a rational function in q^{-s}, where q is the cardinality of the residue field of F. In particular, it has a meromorphic continuation to the entire complex plane. Let $\mathcal{I}$ be the fractional ideal of the ring $\mathbb{C}[q^s, q^{-s}]$ in its fraction field generated by

$$\{Z(\chi, f, f^\vee, \varphi_s) \mid f \in \pi, f^\vee \in \pi^\vee, \varphi_s \text{ is good}\}.$$

Then, $\mathcal{I}$ is generated by $P(q^{-s})^{-1}$, for a unique polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$. We let

$$L(s+\frac{1}{2},\pi,\chi) = \frac{1}{P(q^{-s})}$$

be the local doubling L-series of Piatetski-Shapiro and Rallis. Same construction can also be applied to the archimedean case.

Now suppose E/F is unramified (including split) at v and ψ, χ, π are also unramified. Let $f_0 \in \pi^{\mathcal{K}'}$, $f_0^\vee \in (\pi^\vee)^{\mathcal{K}'}$ such that $\langle f_0, f_0^\vee \rangle = 1$, and φ_s^0 be the unramified standard section. Then the calculation in [GPSR1987] and [Li1992] (see [Li1992, Theorem 3.1]) shows that

$$Z(\chi, f_0, f_0^\vee, \varphi_s^0) = \frac{L(s+\frac{1}{2}, \mathrm{BC}(\pi)\otimes\chi)}{b_{2n}(s)}$$

where

$$b_m(s) = \prod_{i=0}^{m-1} L(2s+m-i, \epsilon_{E/F}^i) \tag{2.7}$$

is a product of local Tate factors. For the general case,

$$\frac{b_{2n}(s)Z(\chi, f, f^\vee, \varphi_s)}{L(s+\frac{1}{2},\pi,\chi)}$$

admits a meromorphic extension to the entire complex plane which is holomorphic at $s = 0$. Moreover, the *normalized zeta integral*

$$Z^*(\chi, f, f^\vee, \varphi_s) = \frac{b_{2n}(s)Z(\chi, f, f^\vee, \varphi_s)}{L(s+\frac{1}{2},\pi,\chi)}|_{s=0} \tag{2.8}$$

defines a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_{2n}(0,\chi), \pi^\vee \boxtimes \chi\pi)$ (cf. [HKS1996, Proof of Theorem 4.3 (1)]).

Remark 2.2.3. It is conjectured (*cf.* e.g., [HKS1996]) that for all irreducible admissible representations π of H' and characters χ of $E^\times$, we have

$$L(s,\pi,\chi) = L(s, \mathrm{BC}(\pi)\otimes\chi).$$

This is known when E/F, χ and π are all unramified due to (the same method of) Kudla–Rallis [KR2005, Section 5]. It is also known when $n = 1$ due to [Har1993].

For further discussion, we need to recall a result on degenerate principal series. In what follows, we will use the notation H'' instead of $\mathrm{U}(W'')$ for short, and recall our embedding $\imath : H' \times H' \hookrightarrow H''$. Let V be a hermitian space of dimension $2n$ over E. Then $\varphi_\phi(g) = \omega_\chi(g)\phi(0)$ defines an H''-intertwining map $\mathcal{S}(V^{2n}) \to I_{2n}(0,\chi)$ whose image $R(V,\chi)$ is isomorphic to $\mathcal{S}(V^{2n})_H$, the maximal H-invariant quotient of $\mathcal{S}(V^{2n})$. Recall in 2.1.3 that we denote by $V^\pm$ the two non-isometric hermitian spaces of dimension $2n$ when v is finite nonsplit; V^+ the only hermitian space (up to isometry) of dimension $2n$ when v is finite split; and V_s $(0 \le s \le 2n)$ the $2n+1$ non-isometric hermitian spaces of dimension $2n$ when v is infinite.

Proposition 2.2.4. *Let notations be as above. We have*

1. *If v is finite nonsplit, then $R(V^+,\chi)$ and $R(V^-,\chi)$ are irreducible, inequivalent, and $I_{2n}(0,\chi) = R(V^+,\chi) \oplus R(V^-,\chi)$.*
2. *If v is finite split, then $R(V,\chi)$ is irreducible, and $I_{2n}(0,\chi) = R(V^+,\chi)$.*
3. *If v is infinite, then $R(V_s,\chi)$ are irreducible, inequivalent, and $I_{2n}(0,\chi) = \bigoplus_{s=0}^{2n} R(V_s,\chi)$.*

Proof. 1. It is [KS1997, Theorem 1.2].

2. It is [KS1997, Theorem 1.3].

3. It is [Lee1994, Section 6, Proposition 6.11].

□

2.3 Central special values of L-functions

2.3.1 Theta lifting and central L-values

We study the relation between the theta lifting θ_ϕ^f defined in 2.2.1 and the central special value of the L-function of the representation π.

Recall that we have an irreducible unitary cuspidal automorphic representation π of $H' = H_n$ and a hermitian space V over E of dimension $2n$. One key question in the theory of theta lifting is whether θ_ϕ^f is nonvanishing. A sufficient condition is to look at the local invariant functional as follows. We have the following proposition, which is usually referred as *theta dichotomy*.

Proposition 2.3.1. *For every nonsplit place $v \in \Sigma$, $\mathrm{Hom}_{H'_v \times H'_v}(R(V_v,\chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v) \neq 0$ for exactly one hermitian space V_v (up to isometry) over E_v of dimension $2n$.*

Such V_v will be denoted as $V(\pi_v,\chi_v)$.

Proof. If v is (real) archimedean, it is due to [Pau1998, Theorem 2.9]. If v is non-archimedean, it is due to Proposition A.1.1 or [GG2011, Theorem 2.10], and the nonvanishing of Z^*. □

In Proposition 2.3.1, if we let $\varphi_s = \varphi_{\phi\otimes\phi^\vee,s}$ and set

$$Z^*(s, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee) = Z^*(\chi_v, f_v, f_v^\vee, \varphi_{\phi_v\otimes\phi_v^\vee,s}),$$

then both sides have meromorphic continuation to the entire complex plane, which are holomorphic at the point $s = 0$. Namely, we have

$$\langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H = \frac{L(\frac{1}{2}, \pi, \chi)}{2\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v\in S} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee), \tag{2.9}$$

in which the product of normalized zeta integrals can actually be taken over a finite set S by the unramified calculation. In particular, for $v \notin S$, $V_v \cong V(\pi_v, \chi_v)$. Then one necessary condition for θ_ϕ^f to be nonvanishing for some f and ϕ is that each local (normalized) zeta integral is not identically zero. In other words, we should have $V_v \cong V(\pi_v, \chi_v)$ for all $v \in \Sigma$. Let $\mathbb{V}(\pi, \chi)$ be the hermitian space over $\mathbb{A}_E$ such that $\mathbb{V}(\pi, \chi)_v \cong V(\pi_v, \chi_v)$, and let $\epsilon(\pi_v, \chi_v) = \epsilon(V(\pi_v, \chi_v))$, $\epsilon(\pi, \chi) = \prod_{v\in\Sigma} \epsilon(\pi_v, \chi_v)$. If $\epsilon(\pi, \chi) = -1$. Then $\mathbb{V}(\pi, \chi)$ is incoherent, and hence for every V, the theta lifting θ_ϕ^f always vanishes. If $\epsilon(\pi, \chi) = 1$, then $\mathbb{V}(\pi, \chi) \cong V(\pi, \chi) \otimes_F \mathbb{A}_F$ for a unique hermitian space $V(\pi, \chi)$ (up to isometry) over E. Assume that $V(\pi, \chi)$ is anisotropic. Then there exist some $f \in \pi$ and $\phi \in \mathcal{S}(V(\pi, \chi)^n)$ such that $\theta_\phi^f \neq 0$ if and only if $L(\frac{1}{2}, \pi, \chi) \neq 0$.

We would like to give another interpretation of the formula (2.9) when $\epsilon(\pi, \chi) = 1$, which is heuristic for the arithmetic case. For this purpose, let us assume the following conjecture proposed by S. Kudla and S. Rallis (in the symplectic-orthogonal case).

Conjecture 2.3.2 (*cf.* [HKS1996])**.** *We have*

$$\dim \mathrm{Hom}_{H'_v\times H'_v}(I_{2n}(0, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v) = 1 \tag{2.10}$$

for every irreducible admissible representation π_v of H'_v.

Remark 2.3.3. 1. When $n = 1$, the above conjecture is proved as Proposition A.2.1.

2. In general, the above conjecture follows from the multiplicity preservation (*cf.* [LST2011, Theorem A]) and the Local Howe Duality Conjecture.

3. The multiplicity preservation has been proved by Waldspurger [Wal1990] when v is non-archimedean

and has odd residue characteristic; by Li–Sun–Tian [LST2011] for every non-archimedean place v; and by Howe [How1989] for every archimedean place v.

4. The Local Howe Duality Conjecture has been proved by Waldspurger [Wal1990] when v is non-archimedean and has odd residue characteristic; by Minguez [Mín2008] for type II dual pairs for every non-archimedean place v; and by Howe [How1989] for every archimedean place v.

5. In particular, the above conjecture is known if v does not have residue characteristic 2; or v has residue characteristic 2 but is split in E.

Let $V = V(\pi, \chi)$ and $R(V, \chi) = \bigotimes' R(V_v, \chi_v)$. On the one hand, the functional

$$\beta(f, f^\vee, \phi, \phi^\vee) = \langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H$$

defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A}_F) \times H'(\mathbb{A}_F)}(R(V, \chi), \pi^\vee \boxtimes \chi\pi) = \bigotimes_{v \in \Sigma} \mathrm{Hom}_{H'_v \times H'_v}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v).$$

On the other hand, the functional

$$\alpha(f, f^\vee, \phi, \phi^\vee) = \prod_{v \in \Sigma} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee) \tag{2.11}$$

(when everything is decomposable, otherwise we take the linear combination) defines also an element in $\bigotimes_{v\in\Sigma} \mathrm{Hom}_{H'_v \times H'_v}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$, which is nonzero. However, by (2.10), the space $\mathrm{Hom}_{H'_v \times H'_v}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$ is of dimension one. Therefore, β is a constant multiple of α. This constant, by (2.9), is equal to

$$\frac{\beta}{\alpha} = \frac{L(\frac{1}{2}, \pi, \chi)}{2 \prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)}.$$

In other words, vanishing of $L(\frac{1}{2}, \pi, \chi)$ is the obstruction for β to be a nontrivial global invariant functional. This kind of formulation is first observed by X. Yuan, S.-W. Zhang and W. Zhang in [YZZa, YZZb].

2.3.2 Vanishing of central L-values

We will prove that the central L-value $L(\frac{1}{2}, \pi, \chi)$ vanishes when $\epsilon(\pi, \chi) = -1$.

By Proposition 2.2.4, we have a decomposition of $H''(\mathbb{A}_F)$-admissible representation

$$I_{2n}(0,\chi) = \bigoplus_{\mathbb{V}} R(\mathbb{V},\chi) = \bigoplus_{\mathbb{V}} {\bigotimes}'_{v\in\Sigma} R(\mathbb{V}_v,\chi_v),$$

where the direct sum is taken over all (isometry classes of) hermitian spaces over $\mathbb{A}_E$ of rank $2n$. We note that each $R(\mathbb{V},\chi)$ is irreducible. Recall the group $H'' = \mathrm{U}(W'')$ and its standard parabolic subgroup P with is unipotent radical N as in Notation 2.2.1. First, we need some lemmas on local representations.

Fix a place v and suppress it from notations. For $T \in \mathrm{Her}_{2n}(E)$, let $\mathbf{O}_T = \{x \in V^{2n} \mid T(x) = T\}$. Define a character ψ_T of $N \cong \mathrm{Her}_{2n}(E)$ by $\psi_T(n(b)) = \psi(\mathrm{tr}\, Tb)$.

Lemma 2.3.4. *Let notations be as above. We have*

1. *Suppose v is finite. Let $\mathcal{S}(V^{2n})_{N,\psi_T}$ (resp. $R(V,\chi)_{N,\psi_T}$) be the twisted Jacquet module of $\mathcal{S}(V^{2n})$ (resp. $R(V,\chi)$) associated to N and the character ψ_T. Then*
   (a) *The quotient map $\mathcal{S}(V^{2n}) \to \mathcal{S}(V^{2n})_{N,\psi_T}$ can be realized by the restriction $\mathcal{S}(V^{2n}) \to \mathcal{S}(\mathbf{O}_T)$.*
   (b) *If T is nonsingular, then*
   $$\dim R(V,\chi)_{N,\psi_T} = \begin{cases} 1 & \text{if } \mathbf{O}_T \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$
2. *Suppose v is infinite, i.e. $E/F = \mathbb{C}/\mathbb{R}$ and T is nonsingular. The space of H-invariant tempered distribution D on $\mathcal{S}(V^{2n})$ such that*
   $$D(\omega_\chi(X)\Phi) = \mathrm{d}\psi_T(X) D(\Phi)$$
   for $X \in \mathrm{Lie}\, N$ is of dimension 1 (resp. 0) if $\mathbf{O}_T \neq \emptyset$ (resp. $\mathbf{O}_T = \emptyset$).

Proof. 1. It is [Ral1987, Lemma 4.2].

2. It is [Ral1987, Lemma 4.2] and [KR1994, Proposition 2.9].

□

We now construct the twisted Jacquet module $R(V,\chi)_{N,\psi_T}$ or the invariant distribution explicitly if it is not trivial. For a standard section $\varphi_s \in I_{2n}(s,\chi)$, define the *Whittaker integral*

$$W_T(g,\varphi_s) = \int_N \varphi_s(\mathsf{w}ng)\psi_T(n)^{-1}\mathrm{d}n,$$

where $\mathsf{w} = \mathsf{w}_{2n}$ and $\mathrm{d}n$ is selfdual with respect to ψ. The integral $W_T(g,\varphi_s)$ is absolutely convergent when $\operatorname{Re} s > n$. It is easy to see that $W_T(e,\bullet) : I_{2n}(s,\chi) \to \mathbb{C}_{N,\psi_T}$ is an N-intertwining map. Let $W_T(s,g,\Phi) = W_T(g,\varphi_{\Phi,s})$ for $\Phi \in \mathcal{S}(V^{2n})$. We have the following lemmas.

Lemma 2.3.5. *Assume that T is nonsingular. Then*

1. *$W_T(g,\varphi_s)$ is entire.*
2. *The map $\Phi \mapsto W_T(0,e,\Phi)$ realizes the surjective composed N-intertwining map $\mathcal{S}(V^{2n}) \to R(V,\chi) \to R(V,\chi)_{N,\psi_T}$ or the invariant distribution in Lemma 2.3.4 (2).*

Proof. 1. It is [Kar1979, Corollary 3.6.1] for v finite and [Wal1988, Theorem 8.1] for v infinite.

2. It is [KR1994, Proposition 2.7].

□

Lemma 2.3.6. *Assume that v is a finite place; E/F, ψ and χ are all unramified; and $V = V^+$. Let Φ^0 be the characteristic function of $(\Lambda^+)^{2n}$ for a selfdual $\mathcal{O}_E$-lattice Λ^+ of V, and $T \in \mathrm{Her}_{2n}(\mathcal{O}_F)$ with $\det T \in \mathcal{O}_F^\times$. Then we have*

$$W_T(s,e,\Phi^0) = b_{2n}(s)^{-1}.$$

Proof. This is [Tan1999, Proposition 3.2]. □

Now suppose we are in the global situation. We denote by $\mathcal{A}(G)$ the space of automorphic forms of G for a reductive group G. For $T \in \mathrm{Her}_{2n}(F)$, define the T-th Fourier coefficient of an $f(g) \in \mathcal{A}(H'')$ as

$$W_T(g,f) = \int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)\psi_T(n)^{-1}\mathrm{d}n.$$

For a hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank $2n$, we have a family of linear maps

$$\begin{aligned} \mathscr{E}_s : R(\mathbb{V},\chi) &\to \mathcal{A}(H'') \\ \Phi &\mapsto E(s,g,\Phi) = E(g,\varphi_{\Phi,s}) \end{aligned} \tag{2.12}$$

for s near 0. It is an $H''(\mathbb{A}_F)$-intertwining map exactly when $s = 0$. Then for T nonsingular (and s near 0), we have

$$E_T(s, g, \Phi) := W_T(g, \mathscr{E}_s(\Phi)) = \prod_{v \in \Sigma} W_T(s, g_v, \Phi_v). \tag{2.13}$$

Lemma 2.3.7. *For every $H''(\mathbb{A}_F)$-intertwining map $\mathscr{E} : R(\mathbb{V}, \chi) \to \mathcal{A}(H'')$, if $W_T(g, \bullet) \circ \mathscr{E}$ vanishes for all nonsingular T, then $\mathscr{E} = 0$.*

Proof. Fix a finite place v, by Lemma 2.3.4 (1), we can find a section $\Phi_0 = \Phi_{v,0}\Phi^v \in \mathcal{S}(\mathbb{V}^{2n})$ with nonzero projection in $R(\mathbb{V}, \chi)$ such that $\Phi_{v,0} \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ (*cf.* Definition 2.4.1).

For every $g^v \in e_v H''(\mathbb{A}_F^v)$, the functional $\Phi_v \mapsto W_T(0, g^v, \Phi_v\Phi^v)$ factors through the twisted Jacquet module $\mathcal{S}(\mathbb{V}_v^{2n})_{N_v,\psi_T}$. If T is singular, then by our choice of $\Phi_{v,0}$ and Lemma 2.3.4 (1-a), $W_T(0, g^v, \Phi_{v,0}\Phi^v) = 0$. Similarly, $W_T(0, g, \Phi_{v,0}\Phi^v) = 0$ for all $g \in P_v H''(\mathbb{A}_F^v)$ since the action of P_v stabilizes the subspace $\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. For T nonsingular, $W_T \equiv 0$ by the assumption. Therefore, $\mathscr{E}(\Phi_0)(g) = 0$ for $g \in P_v H''(\mathbb{A}_F^v)$. It follows that $\mathscr{E}(\Phi_0) = 0$ since $H''(F)P_v H''(\mathbb{A}_F^v)$ is dense in $H''(\mathbb{A}_F)$. Therefore, $\mathscr{E} = 0$ by our choice of Φ_0 and the irreducibility of $R(\mathbb{V}, \chi)$. $\square$

Proposition 2.3.8. *We have*

1. *If $\mathbb{V}$ is incoherent, then* $\dim \mathrm{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathcal{A}(H'')) = 0$.

2. *If $\mathbb{V}$ is coherent, then* $\dim \mathrm{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathcal{A}(H'')) = 1$ *and $\mathscr{E}_0$* (2.12) *is a basis.*

Proof.

1. Assume that $\mathscr{E}$ is a nontrivial intertwining map. Then by Lemma 2.3.7, there is a nonsingular $T \in \mathrm{Her}_{2n}(F)$ such that $W_T(g, \bullet) \circ \mathscr{E}$ does not vanish. By Lemma 2.3.4 (1-b) and (2), T is representable by $\mathbb{V}_v$ for every $v \in \Sigma$, *i.e.* $\mathbf{O}_T \neq \emptyset$. However, $\mathbb{V}$ will be coherent, which is a contradiction.

2. Assume that $\mathscr{E}$ and $\mathscr{E}'$ are both nontrivial intertwining maps. By Lemma 2.3.7, there is a nonsingular T such that $W_T(g, \bullet) \circ \mathscr{E}$ does not vanish. By Lemma 2.3.4 (1-b)(2), there exists $c \in \mathbb{C}$ such that $W_T(g, \bullet) \circ \mathscr{E}' = cW_T(g, \bullet) \circ \mathscr{E}$. Furthermore, c is independent of nonsingular T since all of those that can be represented by $\mathbb{V}$ are in a single $M(F)$-orbit under the conjugation action on $N(F)$. By Lemma 2.3.7, $\mathscr{E}' - c\mathscr{E} = 0$, *i.e.* $\dim \mathrm{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathcal{A}(H'')) \leq 1$.

   For the rest, we need to prove that $\mathscr{E}_0$ is actually nontrivial. Since $\mathbb{V}$ is coherent, we can choose a nonsingular $T \in \mathrm{Her}_{2n}(F)$ that is representable by $\mathbb{V}$. By (2.13), Lemma 2.3.5 (2) and Lemma 2.3.6, we can find a suitable Φ such that $W_T(0, e, \Phi) \neq 0$. Therefore, $\mathscr{E}_0 \neq 0$.

$\square$

Now we can state the following main result.

Theorem 2.3.9. *If $\epsilon(\pi,\chi)=-1$, then $L(\frac{1}{2},\pi,\chi)=0$.*

Proof. Let $\mathbb{V}=\mathbb{V}(\pi,\chi)$. Then it is incoherent. We can choose suitable f_v, $f_v^\vee$, ϕ_v and $\phi_v^\vee$ when some of E,ψ,χ,π is ramified at v, such that $Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)\neq 0$. Let f, $f^\vee$, ϕ and $\phi^\vee$ be adèlic vectors with these subscribed local components and unramified ones at the places where E,ψ,χ,π are unramified. From Proposition 2.2.2 (after analytic continuation), we have

$$\begin{aligned}&\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(0,\imath(g_1,g_2),\phi\otimes\phi^\vee)\mathrm{d}g_1\mathrm{d}g_2\\ =&\frac{L(\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee).\end{aligned}$$

However, $\mathscr{E}_0$ is zero on $R(\mathbb{V},\chi)$ by Proposition 2.3.8 (1). We have $E(0,\imath(g_1,g_2),\phi\otimes\phi^\vee)\equiv 0$. Therefore, $L(\frac{1}{2},\pi,\chi)=0$ by our choices and the fact that the Tate L-values appearing above are finite. □

Since $L(\frac{1}{2},\pi,\chi)=0$, it motivates us to consider its derivative at $\frac{1}{2}$. In fact, we have

$$\begin{aligned}&\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\,E(s,\imath(g_1,g_2),\phi\otimes\phi^\vee)\mathrm{d}g_1\mathrm{d}g_2\\ =&\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(s,\imath(g_1,g_2),\phi\otimes\phi^\vee)\mathrm{d}g_1\mathrm{d}g_2\\ =&\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\frac{L(s+\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(2s+i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(s,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)\\ =&\frac{L'(\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)+L(\frac{1}{2},\pi,\chi)\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\frac{\prod_{v\in S}Z^*(s,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)}{\prod_{i=1}^{2n}L(2s+i,\epsilon_{E/F}^i)}\\ =&\frac{L'(\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee). \qquad (2.14)\end{aligned}$$

Definition 2.3.10 (Analytic kernel functions)**.** We call $E'(0,g,\Phi)=\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\,E(s,g,\Phi)$ the *analytic kernel function* associated to the *test function* $\Phi\in\mathcal{S}(\mathbb{V}^{2n})$.

Recall that for $T\in\mathrm{Her}_{2n}(F)$, we let

$$E_T(s,g,\Phi)=W_T(g,\mathscr{E}_s(\Phi))$$

for s near 0. If T is nonsingular, then

$$W_T(g,\mathscr{E}_s(\Phi))=\prod_{v\in\Sigma}W_T(s,g_v,\Phi_v)$$

if $\Phi = \otimes\Phi_v$ is decomposable. Therefore,

$$E(s, g, \Phi) = \sum_{T \text{ sing.}} E_T(s, g, \Phi) + \sum_{T \text{ nonsing.}} \prod_{v \in \Sigma} W_T(s, g_v, \Phi_v).$$

Taking derivative at $s = 0$, we have

$$\begin{aligned} E'(0, g, \Phi) &= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{T \text{ nonsing.}} \sum_{v \in \Sigma} W'_T(0, g_v, \Phi_v) \prod_{v' \neq v} W_T(0, g_{v'}, \Phi_{v'}) \\ &= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v \in \Sigma} \sum_{T \text{ nonsing.}} W'_T(0, g_v, \Phi_v) \prod_{v' \neq v} W_T(0, g_{v'}, \Phi_{v'}). \end{aligned}$$

However, $\prod_{v' \neq v} W_T(0, g_v, \Phi_v) \neq 0$ only if $\mathbb{V}_{v'}$ represents T for all $v' \neq v$ by Lemma 2.3.4 (1-b). Since $\mathbb{V}$ is incoherent, $\mathbb{V}_v$ can not represent T. For T nonsingular, there are only finitely many $v \in \Sigma$ such that T is not represented by $\mathbb{V}_v$, *i.e.* there does not exist $x_1, \ldots, x_{2n} \in \mathbb{V}_v$ whose moment matrix is T. We denote the set of such v by $\mathrm{Diff}(T, \mathbb{V})$. Then

$$E'(0, g, \Phi) = \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v \in \Sigma} E_v(0, g, \Phi),$$

where

$$E_v(0, g, \Phi) = \sum_{\mathrm{Diff}(T, \mathbb{V}) = \{v\}} W'_T(0, g_v, \Phi_v) \prod_{v' \neq v} W_T(0, g_{v'}, \Phi_{v'}). \tag{2.15}$$

In fact, the summation in (2.15) is taken only over those v that are nonsplit in E.

2.4 Analytic kernel functions

2.4.1 Regular test functions

We prove that the summation of $E'_T(0, g, \Phi)$ over singular T's vanishes for certain choice of Φ, and g in a suitable subset of $H''(\mathbb{A}_F)$. We follow the idea in [YZZb].

Definition 2.4.1 (Regular test functions)**.** Let v be a place and V a hermitian space over E_v. A test function $\phi \in \mathcal{S}(V^r)$ is *regular*, if $\phi(x) = 0$ for x with degenerate moment matrix, *i.e.* $\det T(x) = 0$. We denote by $\mathcal{S}(V^r)_{\mathrm{reg}} \subset \mathcal{S}(V^r)$ the subspace of regular test functions.

Fix a finite subset $S \subset \Sigma_{\mathrm{fin}}$ with $|S| = k > 0$ and let $\mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} = \bigotimes_{v \in S} \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. We have the following proposition.

Proposition 2.4.2. *For $\Phi = \Phi_S \Phi^S \in \mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} \otimes \mathcal{S}(\mathbb{V}^{S,2n})$, $\mathrm{ord}_{s=0} E_T(s, g, \Phi) \geq k$ for T singular and $g \in P(\mathbb{A}_{F,S}) H''(\mathbb{A}_F^S)$.*

We can assume that $\Phi = \otimes_{v \in \Sigma} \Phi_v$ is decomposable with $\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for $v \in S$ and $\mathrm{rank}\, T = 2n - r < 2n$. Choose $a \in \mathrm{GL}_{2n}(E)$ such that

$$aT\,{}^t a^\tau = \begin{pmatrix} \mathbf{0}_r & \\ & \widetilde{T} \end{pmatrix} \tag{2.16}$$

with $\widetilde{T} \in \mathrm{Her}_{2n-r}(E)$. Then

$$E_T(s, g, \Phi) = E_{aT\,{}^t a^\tau}(s, m(a) g, \Phi).$$

Therefore, we can assume that T is of the form (2.16).

First, we need a more explicit formula for the singular coefficient E_T. By definition, for $\mathrm{Re}\, s > n$

$$\begin{aligned} E_T(s, g, \Phi) &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash H''(F)} \varphi_{\Phi,s}(\gamma n g) \psi_T(n)^{-1} \mathrm{d}n \\ &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash H''(F)} \left(r(g)\varphi_{\Phi,s}\right)(\gamma n) \psi_T(n)^{-1} \mathrm{d}n \end{aligned} \tag{2.17}$$

where $r(g)$ standards for the action of H'' on $I_{2n}(s, \chi)$ by right translation. We need to unfold this summation. Recall that for $0 \leq d \leq 2n$,

$$\mathsf{w}_{2n,d} = \begin{pmatrix} \mathbf{1}_d & & & \\ & & & \mathbf{1}_{2n-d} \\ & & \mathbf{1}_d & \\ & -\mathbf{1}_{2n-d} & & \end{pmatrix} \tag{2.18}$$

in Notation 1.0.5. Then $\{\mathsf{w}_{2n,d} \mid 0 \leq d \leq 2n\}$ is a set of representatives of the double coset $W_{P/N}\backslash W_{H''}/W_{P/N}$ of Weyl groups. In particular, $\mathsf{w}_{2n,0} = \mathsf{w}_{2n} = \mathsf{w}$. We have a Bruhat decomposition

$$H''(F) = \coprod_{d=0}^{2n} P(F) \mathsf{w}_{2n,d} P(F),$$

where F can be a global field or its local completions.

Lemma 2.4.3. *For $v \in S$ and $g_v \in P_v$, the support of $r(g_v)\varphi_{\Phi_v,s}$ is contained in $P(F_v)\mathsf{w} N(F_v)$.*

Proof. It suffices to prove that $\varphi_{\Phi_v,s}$ vanishes on $P(F_v)\mathsf{w}_{2n,d}P(F_v)$ for $d>0$ since $g_v \in P(F_v)$. For $g=n(b_1)m(a_1)\mathsf{w}_{2n,d}n(b_2)m(a_2) \in P(F_v)\mathsf{w}_{2n,d}P(F_v)$, we have

$$\begin{aligned}\varphi_{\Phi_v,s}(g) &= \left(\omega_{\chi_v}(g)\Phi\right)(0)\lambda(g)^s \\ &= \chi_v(\det a_1a_2)|\det a_1a_2|_{E_v}^n\lambda(g)^s\int_{\mathbb{V}_v^{2n-d}}\psi_{b_2}(T(x))\Phi_v(xa_2)\mathrm{d}x\end{aligned}$$

where $\mathbb{V}_v^{2n-d}$ is viewed as a subset of $\mathbb{V}_v^{2n}$ by the map $(x_1,\dots,x_{2n-d})\mapsto(0,\dots,0,x_1,\dots,x_{2n-d})$. Since Φ_v is regular and $d>0$, $\Phi_v(xa_2)=0$ for $x\in\mathbb{V}_v^{2n-d}$. Therefore, the lemma follows. $\square$

By the above lemma, we have for $g\in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$,

$$\begin{aligned}(2.17) &= \int_{N(F)\backslash N(\mathbb{A}_F)}\sum_{\gamma\in P(F)\backslash P(F)wP(F)}\left(r(g)\varphi_{\Phi,s}\right)(\gamma n)\psi_T(n)^{-1}\mathrm{d}n \\ &= \int_{N(F)\backslash N(\mathbb{A}_F)}\sum_{\gamma\in wN(F)}\left(r(g)\varphi_{\Phi,s}\right)(\gamma n)\psi_T(n)^{-1}\mathrm{d}n \\ &= \int_{N(\mathbb{A}_F)}\left(r(g)\varphi_{\Phi,s}\right)(\mathsf{w}n)\psi_T(n)^{-1}\mathrm{d}n \\ &= \int_{N(\mathbb{A}_F)}\varphi_s(\mathsf{w}n)\psi_T(n)^{-1}\mathrm{d}n \\ &= \prod_{v\in\Sigma}\int_{N_v}\varphi_{v,s}(\mathsf{w}n_v)\psi_T(n_v)^{-1}\mathrm{d}n_v \end{aligned}\tag{2.19}$$

where we denote by φ_s instead of $r(g)\varphi_{\Phi,s}$ for simplicity. Let $S'\subset\Sigma$ be a finite subset containing all infinite places such that for all $v\notin S'$, v', χ_v and ψ_v are unramified; $\varphi_{v,s}$ is the (unique) unramified section in $I_{2n}(s,\chi_v)$ (hence $S'\supset S$) and $\det\widetilde{T}\in\mathcal{O}_{F_v}^\times$. Then

$$(2.19) = \left(\prod_{v\in S'}W_T(e,\varphi_{v,s})\right)W_T(e,\varphi_s^{S'}).\tag{2.20}$$

By [KR1994, Page 36] and [Tan1999, Proposition 3.2],

$$W_T(e,\varphi_s^{S'}) = \frac{a_{2n}^{S'}(s)}{a_{2n-r}^{S'}(s-\frac{r}{2})b_{2n}^{S'}(s)}$$

where

$$a_{m,v}(s) = \prod_{i=0}^{m-1}L_v(2s+i-m+1,\epsilon_{E/F}^i)$$

and

$$b_{m,v}(s)=\prod_{i=0}^{m-1}L_v(2s+m-i,\epsilon_{E/F}^i)$$

as (2.7). Therefore, $W_T(e,\varphi_s^{S'})$ has a meromorphic continuation to the entire complex plane. For $v\in S'$, we normalize the Whittaker integral to be

$$W_T^*(e,\varphi_{v,s})=\frac{a_{2n-r,v}(s-\frac{r}{2})b_{2n,v}(s)}{a_{2n,v}(s)}W_T(e,\varphi_{v,s}).$$

From the argument (and also notations) in [KR1994, Page 35],

$$W_T(e,\varphi_{v,s})=W_{\widetilde{T}}(e,i^*\circ U_{r,v}(s)\varphi_{v,s}).$$

By [PSR1987, Section 4], the (local) intertwining operator $U_{r,v}(s)$ has a meromorphic continuation to the entire complex plane. By Lemma 2.3.5 (1), $W_T(e,\varphi_{v,s})$ and hence $W_T^*(e,\varphi_{v,s})$ have meromorphic continuation to the entire complex plane. Together with the meromorphic continuation of W_T away from S' and W_T^* in S', (2.20) has a meromorphic continuation which equals

$$\frac{a_{2n}(s)}{a_{2n-r}(s-\frac{r}{2})b_{2n}(s)}\prod_{v\in S'}W_T^*(e,\varphi_{v,s}).$$

Proof of Proposition 2.4.2. At the point $s=0$, we have $b_{2n}(0)=\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)\in\mathbb{C}^\times$ and

$$\frac{a_{2n}(0)}{a_{2n-r}(-\frac{r}{2})}=\prod_{i=0}^{r-1}L(-i,\epsilon_{E/F}^{i+1})\in\mathbb{C}^\times.$$

Let $\kappa_v=\mathrm{ord}_{s=0}W_T^*(e,\bullet)$ be the order of the functional at $s=0$ for $v\in S'$ and

$$\kappa_v'=\mathrm{ord}_{s=0}W_T^*(s,e,\bullet)|_{\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}}$$

for $v\in S$. Since $E_T(e,\varphi_\Phi)=0$ if $\Phi=\otimes\Phi_v$ for at least one Φ_v regular, by (2.20) and the proof of Lemma 2.3.7, we have $\kappa_{v_0}'+\sum_{v_0\neq v\in S'}\kappa_v\geq 1$ for every v_0 in S. Also by the definition of W_T, we see that

$$\varphi_{v,0}\mapsto s^{-\kappa_v}W_T^*(e,\varphi_{v,s})|_{s=0}$$

is a nontrivial N-intertwining map from $I_{2n}(0,\chi)$ to $\mathbb{C}_{N,\psi_T}$. Now if $v \in S$, $\varphi_{v,0} = \varphi_{\Phi_v,0}$ for a regular test function $\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. By Lemma 2.3.4 (1-a), $\varphi_{v,0}$ goes to 0 under the above map, *i.e.* $\kappa'_v \geq \kappa_v + 1$ for $v \in S$. Therefore,

$$\mathrm{ord}_{s=0} \prod_{v \in S'} W_T^*(e, \varphi_{v,s}) \geq \sum_{v \in S} \kappa'_v + \sum_{v \in S'-S} \kappa_v \geq k - 1 + \kappa'_{v_0} + \sum_{v_0 \neq v \in S'} \kappa_v \geq k.$$

The proposition follows. $\square$

In conclusion, if we choose S such that $|S| \geq 2$, and $\Phi = \Phi_S \Phi^S \in \mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} \otimes \mathcal{S}(\mathbb{V}^{S,2n})$ which is decomposable, then for $g \in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$, we have

$$E'(0,g,\Phi) = \sum_{v \in \Sigma} E_v(0,g,\Phi). \tag{2.21}$$

2.4.2 Test functions of higher discriminant

We show that if we have a finer choice of Φ_v for $v \in S$, we can even make $W'_T(0,e,\Phi_v) = 0$ for all nonsingular T that are not representable by $\mathbb{V}_v$.

Since the issue is local, we fix a nonsplit place $v \in S$ and suppress it from notations in this subsection. Let V be one of $V^{\pm}$ and V' the other one which is not isometric to V. Let

$$\begin{aligned}
\mathrm{Her}^0_{2n}(E) &= \{T \in \mathrm{Her}_{2n}(E) \mid \det T \neq 0\}\,; \\
\mathcal{H} &= \left\{b \in \mathrm{Her}^0_{2n}(E) \mid b = T(x) \text{ for some } x \in V^{2n}\right\}; \\
\mathcal{H}' &= \left\{b' \in \mathrm{Her}^0_{2n}(E) \mid b' = T(x') \text{ for some } x' \in V'^{2n}\right\}; \\
\mathcal{H}'_d &= \left\{b' + b'' \mid b' \in \mathcal{H}' \text{ and } b'' \in \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})\right\} \bigcap \mathrm{Her}^0_{2n}(E), \qquad d \in \mathbb{Z},
\end{aligned}$$

where $\mathfrak{p}_E$ is the maximal ideal of $\mathcal{O}_E$. Then

- $\mathrm{Her}^0_{2n}(E) = \mathcal{H} \coprod \mathcal{H}'$;
- $\cdots \subsetneq \mathcal{H}'_{-1} \subsetneq \mathcal{H}'_0 \subsetneq \mathcal{H}'_1 \subsetneq \cdots$;
- $\bigcap_d \mathcal{H}'_d = \mathcal{H}'$;
- $\bigcup_d \mathcal{H}'_d = \mathrm{Her}^0_{2n}(E)$.

We say a test function $\Phi \in \mathcal{S}(V^{2n})$ is *of discriminant d* if

$$\{T(x) \mid x \in \mathrm{Supp}(\Phi)\} \bigcap \mathcal{H}'_d = \emptyset,$$

and denote by $\mathcal{S}(V^{2n})_d$ the space of such functions. Set $\mathcal{S}(V^{2n})_{\mathrm{reg},d} = \mathcal{S}(V^{2n})_{\mathrm{reg}} \cap \mathcal{S}(V^{2n})_d$.

Lemma 2.4.4. *For every $d \in \mathbb{Z}$, $\mathcal{S}(V^{2n})_{\mathrm{reg},d}$ is not empty.*

Proof. Fix an element $d \in \mathbb{Z}$. We only need to prove that there exists $T \notin \mathcal{H}'_d$ such that $\det T \neq 0$. Then $\left(T + \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})\right) \cap \mathcal{H}' = \emptyset$. Any test function with support whose elements have moment matrices contained in $\left(T + \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})\right) \cap \mathrm{Her}^0_{2n}(E)$, which is open, will be in $\mathcal{S}(V^{2n})_{\mathrm{reg},d}$. Now we want to find such a T. Take an element $T_1 \in \mathcal{H}$ with $\det T_1 \neq 0$. Since $\mathcal{H}$ is open, we can find a neighborhood $T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{\nu}) \subset \mathcal{H}$ for some $\nu \in \mathbb{Z}$. If $\nu \leq -d$, then we are done. Otherwise, let ϖ be the uniformizer of F. Then $\varpi^{-\nu-d}\left(T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{\nu})\right) \subset \mathcal{H}$. However, $\varpi^{-\nu-d}\left(T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{\nu})\right) = \left(\varpi^{-\nu-d}T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})\right)$. Therefore, $T = \varpi^{-\nu-d}T_1$ will serve for our purpose. $\square$

Since ψ is nontrivial, we can define its *discriminant* d_ψ to be the largest integer d such that the character ψ_T is trivial on $N(\mathcal{O}_F) \cong \mathrm{Her}_{2n}(\mathcal{O}_E)$ for all $T \in \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})$. We slightly abuse notation since d_ψ depends also on $2n$. We also need to mention that this is not the conductor of a p-adic additive character. However, the difference between them depends only on n and the ramification of E/F. We have the following proposition.

Proposition 2.4.5. *Let $d \geq d_\psi$ be an integer and $\Phi \in \mathcal{S}(V^{2n})_{\mathrm{reg},d}$. Then $W_T(s, e, \Phi) \equiv 0$ for $T \in \mathcal{H}'$ nonsingular.*

Proof. For $\mathrm{Re}\, s > n$,

$$W_T(s, e, \Phi) = \int_N \left(\omega_\chi(\mathsf{w}n)\Phi\right)(0)\lambda(\mathsf{w}n)^s \psi_T(n)^{-1}\mathrm{d}n$$

is absolutely convergent. Therefore, it equals

$$\begin{aligned}
&\int_{\mathrm{Her}_{2n}(E)} \left(\int_{V^{2n}} \psi(\mathrm{tr}\, bT(x))\Phi(x)\mathrm{d}x\right) \lambda(\mathsf{w}n(b))^s \psi(-\mathrm{tr}\, Tb)\mathrm{d}b \\
=& \int_{V^{2n}} \Phi(x)dx \int_{\mathrm{Her}_{2n}(E)} \lambda(\mathsf{w}n(b))^s \psi(\mathrm{tr}(T(x) - T)b)\mathrm{d}b \\
=& \int_{V^{2n}} \Phi(x)dx \int_{\mathrm{Her}_{2n}(E)} \lambda(\mathsf{w}n(b))^s \psi_{T(x)-T}(n(b))\mathrm{d}b. \qquad (2.22)
\end{aligned}$$

Since $\lambda(\mathsf{w}n(b)n(b_1)) = \lambda(\mathsf{w}n(b))$ for $b_1 \in \mathrm{Her}_{2n}(\mathcal{O}_E)$,

$$(2.22) = \int_{V^{2n}} \Phi(x)dx \int_{\mathrm{Her}_{2n}(E)/\,\mathrm{Her}_{2n}(\mathcal{O}_E)} \lambda(\mathsf{w}n(b))^s \psi_{T(x)-T}(n(b))db \int_{\mathrm{Her}_{2n}(\mathcal{O}_E)} \psi_{T(x)-T}(n(b_1))\mathrm{d}b_1,$$

in which the last integral is zero for all $x \in \mathrm{Supp}(\Phi)$ by our assumption on Φ. Therefore, $W_T(s, e, \Phi) \equiv 0$ after continuation. In particular, $W_T'(0, e, \Phi) = 0$. $\square$

Remark 2.4.6. Obviously, it is not necessary to assume the dimension of V to be even. All definitions and results above can be applied to arbitrary dimensions.

In conclusion, let S be a finite subset of Σ_{fin} with $|S| \geq 2$, and $\Phi = \otimes\Phi_v \in \mathcal{S}(\mathbb{V}^{2n})$ with $\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for $v \in S$ and $\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg},d_v}$ for $v \in S$ nonsplit with $d_v \geq d_{\psi_v}$. Then combining (2.21), we have

$$E'(0, g, \Phi) = \sum_{v \notin S} E_v(0, g, \Phi) \tag{2.23}$$

for $g \in e_S H''(\mathbb{A}_F^S)$.

2.4.3 Density of test functions

We have made particular choices of test functions to simplify the formula of the analytic kernel function. However, for our proof of the main theorem, arbitrary choices will not work. We now show that there are "sufficiently many" test functions satisfying these choices we have made, in the sense of Proposition 2.4.10. We follow the idea in [YZZa].

We keep our notations in 2.4.1 and 2.4.2. In particular, v will be a place in S and suppressed from notations. Recall that we have an H''-intertwining map

$$\mathcal{S}(V^{2n}) \twoheadrightarrow \mathcal{S}(V^{2n})_H \cong R(V, \chi) \hookrightarrow I_{2n}(0, \chi)$$

through the Weil representation ω_χ''. Therefore, we obtain an $H' \times H'$ admissible representation on $\mathcal{S}(V^{2n})$ through the embedding $\imath$ (2.3).

Lemma 2.4.7. *If v is nonsplit, then for every $d \in \mathbb{Z}$ we have*

$$\mathcal{S}(V^{2n})_{\mathrm{reg}} = \omega_\chi''(m(F^\times \mathbf{1}_{2n}))\mathcal{S}(V^{2n})_{\mathrm{reg},d}.$$

Proof. Fix an element $d \in \mathbb{Z}$. For every function $\Phi \in \mathcal{S}(V^{2n})_{\mathrm{reg}}$, $\mathrm{Supp}(\Phi)$ is a compact subset of $\mathcal{H}$. Since $\mathrm{Her}_{2n}^0(E)\backslash\mathcal{H}_d'$ is open and

$$\bigcup_d \left(\mathrm{Her}_{2n}^0(E)\backslash\mathcal{H}_d'\right) = \mathrm{Her}_{2n}^0(E)\backslash\bigcap_d \mathcal{H}_d' = \mathrm{Her}_{2n}^0(E)\backslash\mathcal{H}' = \mathcal{H},$$

the family $\left(\mathrm{Her}^0_{2n}(E)\backslash\mathcal{H}'_d\right)_{d\in\mathbb{Z}}$ is an open covering of $\mathrm{Supp}(\Phi)$, hence has a finite subcover. Therefore, there exists $d_0 \in \mathbb{Z}$ such that $\mathrm{Supp}(\Phi)\cap\mathcal{H}'_{d_0} = \emptyset$. If $d_0 \geq d$, we are done. Otherwise, consider $\Phi' = \omega''_\chi(m(\varpi^{d_0-d}\mathbf{1}_{2n}))\Phi$. Then $\mathrm{Supp}(\Phi')\cap\mathcal{H}'_d = \emptyset$. The lemma follows. ☐

In the rest of this subsection, let $n = 1$. Then $H' = \mathrm{U}(W_1)$.

Lemma 2.4.8. *Let π be an irreducible admissible representation of H' which is not of dimension 1 and $A : \mathcal{S}(V) \to \pi$ a surjective H'-intertwining map, where H' acts on $\mathcal{S}(V)$ through a Weil representation ω. Then for every ϕ with $A(\phi) \neq 0$, there is $\phi' \in \mathcal{S}(V)_{\mathrm{reg}}$ such that $A(\phi') \neq 0$ and $\mathrm{Supp}(\phi') \subset \mathrm{Supp}(\phi)$.*

Proof. Let $f = A(\phi)$, if there exists $n \in N'$ such that $\pi(n)f \neq f$, then

$$A\left(\omega(n)\phi - \phi\right) = \pi(n)f - f \neq 0,$$

However,

$$\left(\omega(n)\phi\right)(x) - \phi(x) = (\psi(bT(x)) - 1)\phi(x)$$

where $n = n(b)$. We see that $\phi' = \omega(n)\phi - \phi \in \mathcal{S}(V)_{\mathrm{reg}}$ and $\mathrm{Supp}(\phi') \subset \mathrm{Supp}(\phi)$. If for every $n \in N'$, $\pi(n)f = f$, then f will be fixed by an open subgroup of H' containing N since π is smooth. However, any such subgroup will contain $\mathrm{SU}(W_1)$. Therefore, π factors through $H'/\mathrm{SU}(W_1) = \mathrm{U}(W_1)/\mathrm{SU}(W_1) \cong E^{\times,1}$, which contradicts the assumption on π. ☐

Lemma 2.4.9. *Let π_1 and π_2 be two irreducible admissible representations of H' which are not of dimension 1. Then for every surjective $H'\times H'$-intertwining map $B : \mathcal{S}(V)\otimes\mathcal{S}(V) = \mathcal{S}(V^2) \to \pi_1\boxtimes\pi_2$ where $H'\times H'$ acts on $\mathcal{S}(V)\otimes\mathcal{S}(V)$ by a pair of Weil representations $\omega_1\boxtimes\omega_2$, there is an element $\Phi = \phi_1\otimes\phi_2 \in \mathcal{S}(V^2)_{\mathrm{reg}}$ such that $\phi_\alpha \in \mathcal{S}(V)_{\mathrm{reg}}$ ($\alpha = 1, 2$) and $B(\Phi) \neq 0$.*

Proof. Let $\Phi' \in \mathcal{S}(V^2)$ be an element such that $B(\Phi') \neq 0$, write $\Phi' = \sum\phi_{i,1}\otimes\phi_{i,2}$ as an element in $\mathcal{S}(V)\otimes\mathcal{S}(V)$. Therefore, we can assume that there is $\phi_1\otimes\phi_2$ such that $B(\phi_1\otimes\phi_2) \neq 0$. By Lemma 2.4.8, we can also assume that $\phi_1 \in \mathcal{S}(V)_{\mathrm{reg}}$. For $x \in \mathrm{Supp}(\phi_1)$, let V_x be the subspace of V generated by x and V^x its orthogonal complement. They are both nondegenerate hermitian spaces of dimension 1. As H'-representation, $\mathcal{S}(V) = \mathcal{S}(V_x)\otimes\mathcal{S}(V^x)$. Write $\phi_2 = \sum\phi_{i,x}\otimes\phi^x_i$ according to this decomposition. We can assume that there is one $\phi_x\otimes\phi^x$ such that $B(\phi_1\otimes(\phi_x\otimes\phi^x)) \neq 0$. Since as H'-representation, $\mathcal{S}(V^x)$ is generated by the subspace $\mathcal{S}(V^x)_{\mathrm{reg}}$. We can then write

$$\phi_x\otimes\phi^x = \sum\omega_2(g_j)\left(\omega_2^{-1}(g_j)\phi_x\otimes\phi^x_j\right)$$

with $\phi_j^x \in \mathcal{S}(V^x)_{\mathrm{reg}}$. So we can further assume that $B(\phi_1 \otimes (\phi_x \otimes \phi^x)) \neq 0$ with $\phi^x \in \mathcal{S}(V^x)_{\mathrm{reg}}$, *i.e.* $\mathrm{Supp}(\phi_x \otimes \phi^x) \cap V_x = \emptyset$. Applying Lemma 2.4.8 again, we can further assume that there exists $\phi_2^{(x)} \in \mathcal{S}(V)_{\mathrm{reg}}$ such that $\mathrm{Supp}(\phi_2^{(x)}) \subset \mathrm{Supp}(\phi_x \otimes \phi^x)$ and $B(\phi_1 \otimes \phi_2^{(x)}) \neq 0$. The condition that $\mathrm{Supp}(\phi_2) \cap V^x = \emptyset$ is open for x. Therefore, we can find a neighborhood U_x of x such that $(\phi_1 \mid U_x) \otimes \phi_2^{(x)} \in \mathcal{S}(V^2)_{\mathrm{reg}}$. Since $\mathrm{Supp}(\phi_1)$ is compact, we can find Φ of this kind such that $B(\Phi) \neq 0$. □

Recall the (normalized) zeta integrals (2.8), and that for $\Phi \in \mathcal{S}(V^{2n})$, we set $Z^*(s, \chi, f, f^\vee, \Phi) = Z^*(\chi, f, f^\vee, \varphi_{\Phi,s})$. Combining Lemma 2.4.7 and Lemma 2.4.9, we have the following proposition.

Proposition 2.4.10. *Let $n = 1$, $v \in \Sigma_{\mathrm{fin}}$, π be an irreducible cuspidal automorphic representation of H' and $V_v = V(\pi_v, \chi_v)$. For every $d \in \mathbb{Z}$, we can find $f_v \in \pi_v$, $f_v^\vee \in \pi_v^\vee$ and $\phi_\alpha \in \mathcal{S}(V)_{\mathrm{reg}}$ ($\alpha = 1, 2$) with the property that $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(V_v^2)_{\mathrm{reg},d}$ (resp. $\mathcal{S}(V_v^2)_{\mathrm{reg}}$) if v is nonsplit (resp. split) in E, such that the (normalized) zeta integral $Z^*(0, \chi_v, f_v, f_v^\vee, \phi_{1,v} \otimes \phi_{2,v}) \neq 0$.*

Chapter 3

Arithmetic theta lifting and arithmetic kernel functions

In this chapter, we study the geometric part of the arithmetic inner product. In 3.1, we introduce the Shimura varieties of unitary groups which we work with, and their the special cycles and generating series. We prove the theorem of modularity of the generating series. For the case where Shimura varieties are not proper, we need to consider their compactifications, which will be discussed in 3.2. In 3.3, we define the arithmetic theta lifting and prove its cohomological triviality, which enables us to formulate the explicit conjecture for the arithmetic inner product formula. Finally, we restrict ourselves to the situation where $n = 1$ and hence the Shimura varieties are unitary Shimura curves, in 3.4. We review the theory of Néron–Tate height pairing, based on which we define the arithmetic kernel functions and decompose them into local terms.

We fix an additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ such that for every $\iota \in \Sigma_\infty$, $\psi_\iota(t) = \exp(2\pi i t)$ $(t \in F_\iota = \mathbb{R})$ *until the end of this article.*

3.1 Modularity of generating series

3.1.1 Shimura varieties of unitary groups

We recall the notion of Shimura varieties attached to unitary groups. Let $m \geq 2$ and $1 \leq r < m$ be integers. Let $\mathbb{V}$ be a totally positive definite incoherent hermitian space over $\mathbb{A}_E$ of rank m. Let $\mathbb{H} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathrm{U}(\mathbb{V})$ be the unitary group, which is a reductive group over $\mathbb{A}$, and $\mathbb{H}_{\mathrm{der}} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathrm{SU}(\mathbb{V})$ its derived subgroup. Let $\mathbb{T} \cong \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathbb{A}_E^{\times,1}$ be the maximal abelian quotient of $\mathbb{H}$, which is also isomorphic to the center of $\mathbb{H}$. Let $T \cong \mathrm{Res}_{F/\mathbb{Q}} E^{\times,1}$ be the unique (up to isomorphism) $\mathbb{Q}$-torus such that $T \times_{\mathbb{Q}} \mathbb{A} \cong \mathbb{T}$. Here, $E^{\times,1} = \ker(\mathrm{Nm} : E^\times \to F^\times)$, and $\mathbb{A}_E^{\times,1} = \ker\left(\mathrm{Nm} : \mathbb{A}_E^\times \to \mathbb{A}_F^\times\right)$. Then T has the property that $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_{\mathrm{fin}})$. For every open compact subgroup K of $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})$, there is a Shimura variety $\mathrm{Sh}_K(\mathbb{H})$ of dimension $m-1$ defined over the reflex field E. For every embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ over $\iota \in \Sigma_\infty$, we have the following ι°-adic uniformization

$$\mathrm{Sh}_K(\mathbb{H})^{\mathrm{an}}_{\iota^\circ} \cong H^{(\iota)}(\mathbb{Q}) \backslash \left(\mathcal{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_{\mathrm{fin}})/K\right).$$

We briefly explain the notations and meanings above:

- $\mathrm{Sh}_K(\mathbb{H})^{\mathrm{an}}_{\iota^\circ}$ denotes the complex analytification of $\mathrm{Sh}_K(\mathbb{H})$ at the place ι°.
- Let $V^{(\iota)}$ be the *nearby E-hermitian space* of $\mathbb{V}$ at ι, *i.e.* $V^{(\iota)}$ is the unique E-hermitian space (up to isometry) such that $V^{(\iota)}_v \cong \mathbb{V}_v$ for $v \neq \iota$ but $V^{(\iota)}_\iota$ is of signature $(m-1,1)$. Then $H^{(\iota)} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V^{(\iota)})$.
- $\mathcal{D}^{(\iota^\circ)}$ is the symmetric hermitian domain consisting of all negative $\mathbb{C}$-lines in $V^{(\iota)}_\iota$ whose complex structure is given by the action of $F_\iota \otimes_F E \cong \mathbb{R} \otimes_F E$, which is identified with $\mathbb{C}$ via ι.
- The group $H^{(\iota)}(\mathbb{Q})$ diagonally acts on $\mathcal{D}^{(\iota^\circ)}$ by conjugation, and on $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})/K$ by identifying $H^{(\iota)}(\mathbb{A}_{\mathrm{fin}})$ and $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})$ through the corresponding hermitian spaces.

In fact, the underlying real symmetric domain of $\mathcal{D}^{(\iota^\circ)}$ can be identified with the $H^\iota(\mathbb{R})$-conjugacy class of the Hodge map $h^{(\iota)} : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}} \to H^{(\iota)}_{\mathbb{R}} \cong \mathrm{U}(m-1,1)_{\mathbb{R}} \times \mathrm{U}(m,0)^{d-1}_{\mathbb{R}}$ given by

$$h^{(\iota)}(z) = \left(\begin{pmatrix} \mathbf{1}_{m-1} & \\ & \overline{z}/z \end{pmatrix}, \mathbf{1}_m, \ldots, \mathbf{1}_m \right).$$

Assumption 3.1.1. From now on, we will assume that K is sufficiently small, e.g., contained in the principal congruence subgroup for $N \geq 3$ for the natural embedding into the general linear group.

Then $\mathrm{Sh}_K(\mathbb{H})$ is a quasi-projective nonsingular E-scheme. It is proper if and only if

1. $F \neq \mathbb{Q}$; *or*

2. $F = \mathbb{Q}$, $m = 2$ and $\Sigma(\mathbb{V}) \supsetneq \Sigma_\infty$.

The set of geometric connected components of $\mathrm{Sh}_K(\mathbb{H})$ can be identified with $T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathrm{fin}})/\det K$. For every other open compact subgroup $K' \subset K$, we have an étale covering map

$$\pi_K^{K'} : \mathrm{Sh}_{K'}(\mathbb{H}) \to \mathrm{Sh}_K(\mathbb{H}). \tag{3.1}$$

Let $\mathrm{Sh}(\mathbb{H})$ be the projective system $\{\mathrm{Sh}_K(\mathbb{H})\}_K$. On each $\mathrm{Sh}_K(\mathbb{H})$, we have a Hodge bundle $\mathscr{L}_K \in \mathrm{Pic}(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{Q}}$ which is ample. They are compatible under the pullback of $\pi_K^{K'}$, hence define an element

$$\mathscr{L} \in \mathrm{Pic}(\mathrm{Sh}(\mathbb{H}))_{\mathbb{Q}} := \varinjlim_K \mathrm{Pic}(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{Q}}.$$

3.1.2 Special cycles and generating series

Let V_1 be an E-subspace of $\mathbb{V}_{\mathrm{fin}}$, we say V_1 is *admissible* if $(-,-)|_{V_1\times V_1}$ takes values in E, and for every nonzero $x \in V_1$, (x,x) is totally positive. We have the following lemma.

Lemma 3.1.2. *The E-subspace V_1 is admissible if and only if $(-,-)|_{V_1\times V_1}$ is totally positive definite, and for every $\iota \in \Sigma_\infty$, there is an element $h \in \mathbb{H}_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}})$ such that $hV_1 \subset V^{(\iota)} \subset \mathbb{V}_{\mathrm{fin}}$.*

Proof. The "if" direction is obvious. For the "only if" direction, let us assume that V_1 is admissible and fix an arbitrary ι. Take $v_1 \in V_1$ with nonzero norm. Then $q(v_1) = \frac{1}{2}(v_1, v_1)$ is locally a norm for the hermitian form on $V^{(\iota)}$ by the definition of admissibility and the signatures of $\mathbb{V}$ and $V^{(\iota)}$. Thus it is a norm for some $v \in V^{(\iota)}$ by Hasse–Minkowski Theorem. Now we apply Witt Theorem to find an element $h_1 \in \mathrm{U}(\mathbb{V}_{\mathrm{fin}}) = \mathbb{H}(\mathbb{A}_{\mathrm{fin}})$ such that $h_1 v_1 = v$ as elements in $\mathbb{V}_{\mathrm{fin}}$. Choose a vector $v' \in \langle v\rangle^\perp \subset V^{(\iota)}$ with nonzero norm. Let $h' \in \mathbb{H}(\mathbb{A}_{\mathrm{fin}})$ that stabilizes $\langle v'\rangle^\perp$ and acts on the $\mathbb{A}_{\mathrm{fin},E}$-line spanned by v' via the multiplication by $(\det h_1)^{-1}$. Then $h'h_1v_1 = h'v = v$ for $h = h'h_1 \in \mathrm{SU}(\mathbb{V}_{\mathrm{fin}}) = \mathbb{H}_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}})$.

Replacing V_1 by hV_1, we can assume that $v_1 = v \in V_1$. Since $\dim V_1 < m$, we can use induction on r by considering the orthogonal complement of v in V_1 and $V^{(\iota)}$ to find an element $h \in \mathbb{H}_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}})$ such that $hV_1 \subset V^{(\iota)} \subset \mathbb{V}_{\mathrm{fin}}$. $\square$

For admissible V_1, let $\tilde{\mathbb{V}}$ be a totally positive definite (incoherent) hermitian space over $\mathbb{A}_E$ such that $\tilde{\mathbb{V}}_{\mathrm{fin}} \cong V_1^\perp \subset \mathbb{V}_{\mathrm{fin}}$. Let $\tilde{\mathbb{H}}$ be the corresponding unitary group, we have a finite morphism of

Shimura varieties

$$\varsigma^{V_1} : \mathrm{Sh}_{\tilde{K}}(\tilde{\mathbb{H}}) \to \mathrm{Sh}_K(\mathbb{H}) \tag{3.2}$$

where $\tilde{K} = K \cap \tilde{\mathbb{H}}(\mathbb{A}_{\mathrm{fin}})$. Such morphism, under the uniformization at some ι°, sends $(z, \tilde{h}) \in \tilde{\mathcal{D}}^{(\iota^\circ)} \times \tilde{\mathbb{H}}(\mathbb{A}_{\mathrm{fin}})$ to $(z, \tilde{h}h) \in \mathcal{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_{\mathrm{fin}})$, where

- $\tilde{\mathcal{D}}^{(\iota^\circ)}$ consists of elements z in $\mathcal{D}^{(\iota^\circ)}$ satisfying $z \perp hV_1$;
- h is as in Lemma 3.1.2 (with respect to the ι dividing ι°);
- $\tilde{h}$ fixes all elements in hV_1.

The image defines a *special cycle* $Z(V_1)_K \in \mathrm{CH}^r(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{Q}}$. It depends only on the class KV_1.

For $x \in \mathbb{V}_{\mathrm{fin}}^r$, let V_x be the E-subspace of $\mathbb{V}_{\mathrm{fin}}$ generated by the components of x. We define

$$Z(x)_K = \begin{cases} Z(V_x)_K c_1(\mathscr{L}_K^\vee)^{r-\dim_E V_x} & \text{if } V_x \text{ is admissible} \\ 0 & \text{otherwise.} \end{cases}$$

To introduce the generating series, we need a restriction on the space $\mathcal{S}(\mathbb{V}_\iota^r)$ of the Weil representation when ι is infinite. We define a subspace $\mathcal{S}(\mathbb{V}_\iota^r)^{\mathrm{U}_\iota} \subset \mathcal{S}(\mathbb{V}_\iota^r)$, which consists of functions of the form

$$P(T(x)) \exp(-2\pi \operatorname{tr} T(x))$$

where P is a polynomial function on $\mathrm{Her}_r(\mathbb{C})$. Let $\mathcal{S}(\mathbb{V}_\iota^r)^{\mathcal{K}_{r,\iota}\text{-fin}} \subset \mathcal{S}(\mathbb{V}_\iota^r)$ be the subspace of $\mathcal{K}_{r,\iota}$-finite vectors, which is naturally a $(\mathrm{Lie}\, H_{r,\iota}, \mathcal{K}_{r,\iota})$-module. Then $\mathcal{S}(\mathbb{V}_\iota^r)^{\mathrm{U}_\iota}$ is the $(\mathrm{Lie}\, H_{r,\iota}, \mathcal{K}_{r,\iota})$-submodule generated by the Gaussian

$$\phi_\infty^0(x) = \exp(-2\pi \operatorname{tr} T(x)).$$

Let

$$\mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty} = \left(\bigotimes_{\iota \in \Sigma_\infty} \mathcal{S}(\mathbb{V}_\iota^r)^{\mathrm{U}_\iota} \right) \otimes \mathcal{S}(\mathbb{V}_{\mathrm{fin}}^r); \quad \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K} = \left(\bigotimes_{\iota \in \Sigma_\infty} \mathcal{S}(\mathbb{V}_\iota^r)^{\mathrm{U}_\iota} \right) \otimes \mathcal{S}(\mathbb{V}_{\mathrm{fin}}^r)^K$$

for an open compact subgroup K of $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})$.

For $\phi \in \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty\, K}$, we define the *generating series* to be

$$Z_\phi(g) = \sum_{x \in K\backslash \mathbb{V}^r_{\mathrm{fin}}} (\omega_\chi(g)\phi)\,(T(x), x) Z(x)_K$$

as a formal series with values in $\mathrm{CH}^r(\mathrm{Sh}_K(\mathbb{H}))_\mathbb{C}$ for $g \in H_r(\mathbb{A}_F)$. Here for $\phi = \phi_\infty \otimes \phi_{\mathrm{fin}}$, we denote $\phi(T(x), x) = \phi_\infty(y)\phi_{\mathrm{fin}}(x)$ for some $y \in \mathbb{V}^r_\infty$ with $T(y) = T(x)$, whose value does not depend on the choice of y. This makes sense since $Z(x)_K \neq 0$ only for V_x admissible and hence $T(x)$ is totally semi-positive definite. It is easy to see that $Z_\phi(g)$ is compatible under the pullback of $\pi^{K'}_K$, hence defines a formal series with values in $\mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))_\mathbb{C} := \varinjlim_K \mathrm{CH}^r(\mathrm{Sh}_K(\mathbb{H}))_\mathbb{C}$.

3.1.3 Pullback formula and modularity theorem

We introduce the standard symplectic F-space W'_r (comparing to the space W_r in 2.1.1), which has the basis $\{e_1, \ldots, e_{2r}\}$ with the symplectic form

- $\langle e_i, e_j\rangle = 0$;
- $\langle e_{r+i}, e_{r+j}\rangle = 0$;
- $\langle e_i, e_{r+j}\rangle = \delta_{ij}$ for $1 \leq i, j \leq r$.

The group $G_r = \mathrm{Sp}(W'_r)$ is an F-reductive group. Similarly, when defining the generating series, we use the Weil representation ω (with respect to ψ) of $G_r(\mathbb{A}_F) \times \mathbb{G}(\mathbb{A})$ on $\mathcal{S}(\mathbb{V}^r)$.

We briefly recall the notion of Shimura varieties attached to orthogonal groups. The $\mathbb{A}_F$-module $\mathbb{V}$ is also a totally positive definite quadratic space over $\mathbb{A}_F$ of rank $2m$ with the quadratic form $\mathrm{Tr}_{\mathbb{A}_E/\mathbb{A}_F}(-,-)$. It is incoherent with discriminant in $F^\times/F^{\times 2} \subset \mathbb{A}_F^\times/\mathbb{A}_F^{\times 2}$. Let $\mathbb{G} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\,\mathrm{GSpin}(\mathbb{V})$ be the special Clifford group of $\mathbb{V}$ with the adjoint (quotient) group $\mathbb{G}_{\mathrm{adj}} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\,\mathrm{SO}(\mathbb{V})$ and the derived subgroup $\mathbb{G}_{\mathrm{der}} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\,\mathrm{Spin}(\mathbb{V})$. For every open compact subgroup K' of $\mathbb{G}(\mathbb{A}_{\mathrm{fin}})$, there is a Shimura variety $\mathrm{Sh}_{K'}(\mathbb{G})$ defined over the reflex field F such that for every embedding $\iota : F \hookrightarrow \mathbb{C}$, we have the following ι-adic uniformization

$$\mathrm{Sh}_{K'}(\mathbb{G})^{\mathrm{an}}_\iota \cong G^{(\iota)}(\mathbb{Q})\backslash \left(\mathcal{D}'^{(\iota)} \times \mathbb{G}(\mathbb{A}_{\mathrm{fin}})/K'\right).$$

We briefly explain the notations and meanings above:

- $\mathrm{Sh}_{K'}(\mathbb{G})^{\mathrm{an}}_\iota$ denotes the complex analytification of $\mathrm{Sh}_{K'}(\mathbb{G})$ via ι.

- Let $V^{(\iota)}$ be the *nearby F-quadratic space* of $\mathbb{V}$ at ι, *i.e.* $V^{(\iota)}$ is the unique F-quadratic space (up to isometry) such that $V^{(\iota)}_v \cong \mathbb{V}_v$ for $v \neq \iota$ but $V^{(\iota)}_\iota$ is of signature $(2m-2, 2)$. Then $G^{(\iota)} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(V^{(\iota)})$.

- $\mathcal{D}'^{(\iota)}$ is the symmetric hermitian domain consisting of all *oriented* negative definite 2-dimensional subspaces of $V^{(\iota)}_\iota$.

- The group $G^{(\iota)}(\mathbb{Q})$ diagonally acts on $\mathcal{D}'^{(\iota)}$ by conjugation, and on $\mathbb{G}(\mathbb{A}_{\mathrm{fin}})/K'$ by identifying $G^{(\iota)}(\mathbb{A}_{\mathrm{fin}})$ and $\mathbb{G}(\mathbb{A}_{\mathrm{fin}})$ through the corresponding quadratic spaces.

We have similar notions of Hodge bundles, special cycles, and generating series on $\mathrm{Sh}_{K'}(\mathbb{G})$, which are denoted by $\mathscr{L}'_{K'}$, $Z'(x)_{K'}$ for $x \in \mathbb{V}^r_{\mathrm{fin}}$, and $Z'_\phi(g')$ for $\phi \in \mathcal{S}(\mathbb{V}^r)^{\mathrm{O}_\infty K'}$ and $g' \in G_r(\mathbb{A}_{\mathrm{fin}})$, respectively (*cf.* [YZZ2009]).

Fix an embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ over ι and suppress them from the notations of nearby objects like $V = V^{(\iota)}$, $H = H^{(\iota)}$, $\mathcal{D} = \mathcal{D}^{(\iota^\circ)}$, etc. Therefore, we have the usual notion of Shimura variety $\mathrm{Sh}_K(H, X)$ (resp. $\mathrm{Sh}_{K'}(G, X')$ with a connected component X^+ of X'), which is defined over $\iota^\circ(E)$ (resp. $\iota(F)$). Its neutral component is the connected Shimura variety $\mathrm{Sh}^\circ_K(H_{\mathrm{der}}, \overline{X})$ (resp. $\mathrm{Sh}^\circ_{K'}(G_{\mathrm{der}}, \overline{X^+})$) attached to the connected Shimura datum $(H_{\mathrm{der}}, \overline{X})$ (resp. $(G_{\mathrm{der}}, \overline{X^+})$), which is defined over E_K (resp. $E_{K'}$): a finite abelian extension of $\iota(F)$ in $\mathbb{C}$. The canonical embedding $H_{\mathrm{der}} \hookrightarrow G_{\mathrm{der}}$ (*cf.* Remark 3.1.4 (1) below) of reductive groups and the embedding $\mathcal{D} \hookrightarrow \mathcal{D}'$ by forgetting the E-action define an injective map of connected Shimura data $(H_{\mathrm{der}}, \overline{X}) \to (G_{\mathrm{der}}, \overline{X^+})$ which gives an embedding

$$i_{K'} : \mathrm{Sh}^\circ_K(H_{\mathrm{der}}, \overline{X}) \hookrightarrow \mathrm{Sh}^\circ_{K'}(G_{\mathrm{der}}, \overline{X^+})$$

that is defined over E_K, providing that $K \cap H_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}}) = K' \cap H_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}})$ and K' is sufficiently small. Let $Z(x)^\circ_K$ (resp. $Z'(x)^\circ_{K'}$, $Z_\phi(g)^\circ$, $Z'_\phi(g')^\circ$) be the restriction of $Z(x)_K$ (resp. $Z'(x)_{K'}$, $Z_\phi(g)$, $Z'_\phi(g')$) to the neutral component.

Proposition 3.1.3. *Assume that K' is sufficiently small and $K \cap H_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}}) = K' \cap H_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}})$. Then for $x \in \mathbb{V}_{\mathrm{fin}}$, the pullback of the special divisor $i^*_{K'} Z'(x)^\circ_{K'}$ is the sum of $Z(x_1)^\circ_K$ indexed by the classes x_1 in $K \backslash K'x$, both considered as elements in Chow groups.*

Proof. If $x = 0$, then the only class in $K \backslash K'x$ is $x_1 = 0$; the proposition follows from the compatibility of Hodge bundles under pullbacks induced by maps of (connected) Shimura data.

Now we assume that $\langle x, x \rangle \in E$ and is totally positive. Suppose that $(z, h) \in \mathcal{D} \times H_{\mathrm{der}}(\mathbb{A}_{\mathrm{fin}})$ represents a $\mathbb{C}$-point in the scheme-theoretic intersection $\mathrm{Sh}^\circ_K(H_{\mathrm{der}}, \overline{X}) \cap Z'(x_1)^\circ_{K'}$ for some $x_1 \in$

$K'x$. Let $g \in G(\mathbb{A}_{\text{fin}})$ such that $gx_1 = x_1' \in V \subset \mathbb{V}_{\text{fin}}$. Then $z \perp \gamma x_1'$ for some $\gamma \in G(\mathbb{Q})$ and $h \in \gamma G(\mathbb{A}_{\text{fin}})_{x_1'} gk'$ for some $k' \in K'$, where $G(\mathbb{A}_{\text{fin}})_{x_1'}$ is the subgroup of $G(\mathbb{A}_{\text{fin}})$ fixing x_1'. We show that $\gamma G(\mathbb{A}_{\text{fin}})_{x_1'} gk' \cap H_{\text{der}}(\mathbb{A}_{\text{fin}}) = G(\mathbb{A}_{\text{fin}})_{\gamma x_1'} \gamma gk' \cap H_{\text{der}}(\mathbb{A}_{\text{fin}}) \neq \emptyset$, *i.e.* $G(\mathbb{A}_{\text{fin}})_{\gamma x_1'} \cap H_{\text{der}}(\mathbb{A}_{\text{fin}}) k'^{-1} g^{-1} \gamma^{-1} \neq \emptyset$, which is true by Lemma 3.1.2. Therefore, (z, h) represents a $\mathbb{C}$-point in the special cycle $Z(h^{-1}E\langle \gamma g x_1 \rangle)_K^\circ$ of $\mathrm{Sh}_K^\circ(H_{\text{der}}, \overline{X})$. If we write $h = g_1 \gamma g k'$ with some $g_1 \in G(\mathbb{A}_{\text{fin}})_{\gamma x_1'}$, then

$$h^{-1}E\langle \gamma g x_1 \rangle = E\langle h^{-1} \gamma g x_1 \rangle = E\langle k'^{-1} g^{-1} \gamma^{-1} g_1^{-1} \gamma g x_1 \rangle = E\langle k'^{-1} x_1 \rangle.$$

Therefore, the scheme-theoretic intersection is indexed by the classes x_1 in $K \backslash K'x$. This is also true in the Chow groups, as in [YZZ2009, Proposition 2.4]. □

Remark 3.1.4. We have the following two remarks.

1. The canonical embedding $H_{\text{der}} \hookrightarrow G_{\text{der}}$ is given by the following way. First, we have an embedding $H_{\text{der}} \hookrightarrow H \hookrightarrow G_{\text{adj}}$ by forgetting the E-action on $V = V^{(\iota)}$. Since H_{der} is simply connected, we have a canonical lifting $H_{\text{der}} \hookrightarrow G$. Since H_{der} has no nontrivial abelian quotient, the image is actually contained in G_{der}.

2. In the proof of Proposition 3.1.3, we can still use the adèlic description of the $\mathbb{C}$-points of $\mathrm{Sh}_K^\circ(H_{\text{der}}, \overline{X})$ (resp. $\mathrm{Sh}_{K'}^\circ(G_{\text{der}}, \overline{X^+})$), which is compatible with that of $\mathrm{Sh}_K(\mathbb{H})$ (resp. $\mathrm{Sh}_{K'}(\mathbb{G})$) since H_{der} (resp. G_{der}) is semisimple, of noncompact type and simply-connected.

The group G_r is canonically embedded in H_r by identifying the bases $\langle e_1, \ldots, e_{2r} \rangle$ of W_r' and W_r. Therefore, $\omega_\chi|_{G_r} = \omega$. From Proposition 3.1.3, we have the following corollary.

Corollary 3.1.5. *Let $r = 1$ and K, K' be as in Proposition 3.1.3. Then $i_{K'}^* Z_\phi'(g')^\circ = Z_\phi(g')^\circ$ for $g' \in G_1(\mathbb{A}_F)$ and $\phi \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty K'}$.*

For a linear functional $l \in \mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))_{\mathbb{C}}^*$, we have a complex valued series

$$l(Z_\phi)(g) = \sum_{x \in K \backslash \mathbb{V}_{\text{fin}}^r} (\omega_\chi(g)\phi)(T(x), x) l(Z(x)_K)$$

for every K such that ϕ is invariant under K (which is clearly independent of such choice). The following is the main theorem.

Theorem 3.1.6 (Modularity of the generating series)**.** *We have*

1. *If $l(Z_\phi)(g)$ is absolutely convergent, then it is an automorphic form of H_r. Moreover, the archimedean component of (the representation generated by) $l(Z_\phi)$ is a discrete series representation of weight $(\frac{m+\mathfrak{k}^\chi}{2}, \frac{m-\mathfrak{k}^\chi}{2})$ (cf. Definition 2.1.2 and Notation 2.1.1).*

2. *If $r = 1$, then $l(Z_\phi)(g)$ is absolutely convergent for every l.*

Proof. 1. We proceed as in [YZZ2009, Section 4]. First, we can assume that $\phi = \phi^0_\infty \otimes \phi_{\mathrm{fin}}$ since other cases will follow from the $(\mathrm{Lie}\, H_{r,\iota}, \mathcal{K}_{r,\iota})$-action. Assuming the absolute convergence of $l(Z_\phi)(g)$, we only need to check the automorphy, *i.e.* invariance under left translation of $H_r(F)$, since the weight part is clear.

It is easy to check the invariance under $n(b)$ and $m(a)$. For $b \in \mathrm{Her}_r(E)$, the matrix $bT(x)$ is F-rational if $Z(x)_K \neq 0$. Therefore, $l(Z_\phi)(n(b)g) = l(Z_\phi)(g)$ for all $g \in H_r(\mathbb{A}_F)$. For $a \in \mathrm{GL}_r(E)$, we have $Z(xa)_K = Z(x)_K$, which implies that $l(Z_\phi)(m(a)g) = l(Z_\phi)(g)$.

Since $H_r(F)$ is generated by $n(b)$, $m(a)$ and $\mathsf{w}_{r,r-1}$, we only need to check that $l(Z_\phi)(\mathsf{w}_{r,r-1}g) = l(Z_\phi)(g)$ for all $g \in H_r(\mathbb{A}_F)$. Assuming this for $r = 1$ (which is proved in Lemma 3.1.7 below), we prove for general $r > 1$, following [YZZ2009] and [Zha2009].

We suppress l from notations for simplicity. Then for K sufficiently small, we have

$$\begin{aligned} Z_\phi(\mathsf{w}_{r,r-1}g) &= \sum_{x \in K\backslash \mathbb{V}^r_{\mathrm{fin}}} (\omega_\chi(\mathsf{w}_{r,r-1}g)\phi)\,(T(x), x) Z(x)_K \\ &= \sum_{x \in K\backslash \mathbb{V}^{r-1}_{\mathrm{fin}}} \sum_{y \in K_x\backslash \mathbb{V}_{\mathrm{fin}}} (\omega_\chi(\mathsf{w}_{r,r-1}g)\phi)\,(T(x,y),(x,y)) Z((x,y))_K, \end{aligned} \tag{3.3}$$

where K_x is the stabilizer of x in K. Then

$$(3.3) = \sum_{x \in K\backslash \mathbb{V}^{r-1}_{\mathrm{fin}}} \sum_{y_1 \in K_x\backslash \mathbb{V}^x_{\mathrm{fin}}} \sum_{y_2 \in V_x} (\omega_\chi(\mathsf{w}_{r,r-1}g)\phi)\,(T(x,y_1+y_2),(x,y_1+y_2)) Z((x,y_1+y_2))_K, \tag{3.4}$$

where $\mathbb{V}^x_{\mathrm{fin}}$ is the orthogonal complement of $V_x = E\langle x\rangle$ in $\mathbb{V}_{\mathrm{fin}}$. Recalling the morphism ς^{V_x} in (3.2), we have

$$(3.4) = \sum_{x \in K\backslash \mathbb{V}^{r-1}_{\mathrm{fin}}} \sum_{y_1 \in K_x\backslash \mathbb{V}^x_{\mathrm{fin}}} \sum_{y_2 \in V_x} (\omega_\chi(\mathsf{w}_{r,r-1})(\omega_\chi(g)\phi))\,(T(x,y_1+y_2),(x,y_1+y_2)) \varsigma^{V_x}_* Z(y_1)_{K_x}. \tag{3.5}$$

Applying the case $r=1$ to the special cycle $Z(V_x)_{K_x}$, we have

$$(3.5) = \sum_{x\in K\backslash \mathbb{V}_{\mathrm{fin}}^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_{\mathrm{fin}}^{x}} \sum_{y_2\in V_x} \left(\omega_\chi(\mathsf{w}_{r,r-1})\widehat{\omega_\chi(g)\phi}^{y_1}\right)(T(x,y_1+y_2),(x,y_1+y_2))\varsigma_*^{V_x} Z(y_1)_{K_x}, \tag{3.6}$$

where the superscript y_1 means taking the partial Fourier transformation along the y_1 direction. Applying the Poisson Summation Formula (recall that $\phi_\infty = \phi_\infty^0$ is the Gaussian), we have

$$\begin{aligned}
(3.6) &= \sum_{x\in K\backslash \mathbb{V}_{\mathrm{fin}}^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_{\mathrm{fin}}^{x}} \sum_{y_2\in V_x} \left(\omega_\chi(\mathsf{w}_{r,r-1})\widehat{\omega_\chi(g)\phi}^{y_1,y_2}\right)(T(x,y_1+y_2),(x,y_1+y_2))\varsigma_*^{V_x} Z(y_1)_{K_x} \\
&= \sum_{x\in K\backslash \mathbb{V}_{\mathrm{fin}}^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_{\mathrm{fin}}^{x}} \sum_{y_2\in V_x} \left(\omega_\chi(\mathsf{w}_{r,r-1})\widehat{\omega_\chi(g)\phi}^{y}\right)(T(x,y_1+y_2),(x,y_1+y_2))\varsigma_*^{V_x} Z(y_1)_{K_x} \\
&= \sum_{x\in K\backslash \mathbb{V}_{\mathrm{fin}}^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_{\mathrm{fin}}^{x}} \sum_{y_2\in V_x} (\omega_\chi(g)\phi)(T(x,y_1+y_2),(x,y_1+y_2))\varsigma_*^{V_x} Z(y_1)_{K_x} \\
&= \sum_{x\in K\backslash \mathbb{V}_{\mathrm{fin}}^{r}} (\omega_\chi(g)\phi)(T(x),x)Z(x)_K \\
&= Z_\phi(g).
\end{aligned}$$

2. It follows from the argument in Lemma 3.1.7, Corollary 3.1.5 and [YZZ2009, Theorem 1.3], which uses the result in [KM1990].

□

Lemma 3.1.7. *If $r=1$, then $l(Z_\phi)(\mathsf{w}_1 g) = l(Z_\phi)(g)$ for all $g\in H_1(\mathbb{A}_F)$.*

Proof. We suppress l from notations. Further, we fix an element $\iota^\circ \in \Sigma_\infty^\circ$ over $\iota\in\Sigma_\infty$ and suppress them as in the previous discussion. It is clear that we only need to prove $Z_\phi(\mathsf{w}_1 g) = Z_\phi(g)$ for $g\in G_1(\mathbb{A}_F)$ since $G_1(\mathbb{A}_{\infty,F})\mathcal{K}_{1,\infty} = H_1(\mathbb{A}_{\infty,F})$. As before, we assume that ϕ_∞ is the Gaussian and K is sufficiently small. Recall that $\pi_0(\mathrm{Sh}_K(H,X)_{\iota,\mathbb{C}}) \cong T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathrm{fin}})/\det(K)$. We have the following inclusion

$$\mathrm{CH}^1(\mathrm{Sh}_K(H,X))_{\mathbb{C}} \hookrightarrow \bigoplus_{t\in T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathrm{fin}})/\det K} \mathrm{CH}^1(\mathrm{Sh}_K(H,X)_t)_{\mathbb{C}}, \tag{3.7}$$

where $\mathrm{Sh}_K(H,X)_t$ is the (canonical model of the) corresponding (geometric) connected component. Let $h\in H(\mathbb{A}_{\mathrm{fin}})$ such that $\det h\in t$, and T_h be the Hecke operator. Then $\mathrm{T}_h : \mathrm{Sh}_{K^h}(H,X) \to$

$\mathrm{Sh}_K(H,X)$ induces

$$\mathrm{T}_h^\circ : \mathrm{Sh}_{K^h}^\circ(H_{\mathrm{der}}, \overline{X}) = \mathrm{Sh}_{K^h}(H,X)_1 \xrightarrow{\sim} \mathrm{Sh}_K(H,X)_t \hookrightarrow \mathrm{Sh}_K(H,X),$$

where $K^h = hKh^{-1}$. We have $(\mathrm{T}_h^\circ)^* Z_\phi(g) = Z_\phi(g)^\circ$, which is the image of (3.7) composed with the projection to $\mathrm{CH}^1(\mathrm{Sh}_K(H,X)_t)_{\mathbb{C}}$. Here, $Z_\phi(g)^\circ$ is the generating series on $\mathrm{Sh}_{K^h}^\circ(H_{\mathrm{der}}, \overline{X})$. Now shrinking K^h if necessary such that we can apply Corollary 3.1.5, we have $Z_\phi(g)^\circ = i_{K'}^* Z'_\phi(g)^\circ$ for $g \in G_1(\mathbb{A}_{\mathrm{fin}})$. Applying [YZZ2009, Theorem 1.2 or Theorem 1.3], we conclude that $Z_\phi(\mathsf{w}_1 g)^\circ = Z_\phi(g)^\circ$. The lemma follows by (3.7). $\square$

3.2 Smooth compactification of unitary Shimura varieties

We introduce the canonical smooth compactification of the unitary Shimura varieties if they are not proper, and the compactified generating series on them.

3.2.1 Compactified unitary Shimura varieties

Let $m \geq 2$ be an integer, $E = \mathbb{Q}(\mathbf{j}) \subset \mathbb{C}$ with $\mathrm{Im}\,\mathbf{j} > 0$ and $\mathbf{j}^2 = -D$ for some square-free positive integer D, $\mathcal{O}_E$ its ring of integers and τ the nontrivial Galois involution on E. Let $(V, (-,-))$ be a hermitian space of dimension m over E of signature $(m-1,1)$. If $m=2$, we further assume that $\det V \in \mathrm{Nm}\, E^\times$. Let $H = \mathrm{U}(V)$ be the unitary group, we have the Hodge map $h : \mathbb{S} \to H_{\mathbb{R}} \cong \mathrm{U}(m-1,1)_{\mathbb{R}}$ given by

$$h(z) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \overline{z}/z \end{pmatrix}.$$

Then we have the notion of Shimura variety $\mathrm{Sh}_K(H,h)$ for every open compact subgroup K of $H(\mathbb{A}_{\mathrm{fin}})$. For K sufficiently small, it is smooth, quasi-projective but non-proper over E of dimension $m-1$. Therefore, we need to construct a smooth compactification of $\mathrm{Sh}_K(H,h)$ to do height pairing. When $m=2$, it is easy since we only need to add cusps. When $m=3$ and H is quasi-split, a canonical smooth compactification (even of the integral model) has been constructed in [Lar1992]. In fact, the same construction works in more general cases (just for compactification of the canonical model), namely for every H as above. We should mention that, if the signature of V is (a,b) such that

$a \geq b > 1$ or V is defined over a totally real field other than $\mathbb{Q}$ and indefinite at every archimedean place, then we are not clear whether there exists a canonical smooth compactification (in a suitable sense).

Now let us assume that $m > 2$. In order to use modular interpretations, we should work with the group of unitary similitude. For every $v, w \in V$, the pairing

$$(v, w)' = \mathrm{Tr}_{E/\mathbb{Q}}(\mathbf{j}(v, w))$$

defines a symplectic form of V satisfying $(ev, w)' = (v, e^\tau w)'$ for every $e \in E$. Let $GH = \mathrm{GU}(V)$ such that for every $\mathbb{Q}$-algebra R,

$$GH(R) = \left\{h \in \mathrm{GL}_m(E \otimes_{\mathbb{Q}} R) \mid (hv, hw)' = \lambda(h)(v, w)' \text{ for some } \lambda(h) \in R^\times\right\}$$

and the Hodge map $Gh : \mathbb{S} \to GH_{\mathbb{R}} \cong \mathrm{GU}(m-1, 1)_{\mathbb{R}}$ is given by

$$Gh(z) = \begin{pmatrix} z & & & \\ & \ddots & & \\ & & z & \\ & & & \overline{z} \end{pmatrix}.$$

For every sufficiently small open compact subgroup K of $GH(\mathbb{A}_{\mathrm{fin}})$, we have the Shimura variety $\mathrm{Sh}_K(GH, Gh)$ which is smooth, quasi-projective but non-proper over E of dimension $m-1$. Although we do not have a map of Shimura data, $\mathrm{Sh}_{K \cap H(\mathbb{A}_{\mathrm{fin}})}(H, h)$ and $\mathrm{Sh}_K(GH, Gh)$ have the same neutral component for sufficiently small K. Therefore, it is same to give a canonical smooth compactification of $\mathrm{Sh}_K(GH, Gh)$ instead of the original one. In fact, $\mathrm{Sh}_K(GH, Gh)$ is the moduli space of abelian varieties of certain PEL type. We fix a lattice $V_{\mathbb{Z}}$ of V such that $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^\perp$ and let $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Then $\mathrm{Sh}_K(GH, Gh)$ represents the following moduli functor on the category of locally noetherian E-schemes: for every such scheme S, $\mathrm{Sh}_K(GH, Gh)(S)$ is the set of isomorphism classes of quadruples $(A, \theta, i, \bar{\eta})$, where

- A is an abelian scheme over S of dimension m.
- $\theta : A \to A^\vee$ is a polarization.
- $i : \mathcal{O}_E \hookrightarrow \mathrm{End}_S(A)$ is a monomorphism of rings with units, such that $\mathrm{tr}(i(e); \mathrm{Lie}_S(A)) = (m-1)e + e^\tau$ and $\theta \circ i(e) = i(e^\tau)^\vee \circ \theta$ for all $e \in \mathcal{O}_E$. Here, we view $(m-1)e + e^\tau$ as a constant

section of $\mathscr{O}_S$ via the structure map $E \to \mathscr{O}_S$.

- $\bar{\eta}$ is a K-level structure, that is, for chosen geometric point s on each connected component of S, $\bar{\eta}$ is a $\pi_1(S,s)$-invariant K-class of $\mathcal{O}_E \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$-linear symplectic similitude $\eta : V_{\widehat{\mathbb{Z}}} \to \mathrm{H}_1^{\text{ét}}(A_s, \widehat{\mathbb{Z}})$, where the pairing on the latter space is the θ-Weil pairing (hence the degree of θ is $[V_{\mathbb{Z}}^{\perp} : V_{\mathbb{Z}}]$). In fact, such data are independent of the geometric point s we choose.

In the theory of toroidal compactification (*cf.* [AMRT2010]), we need to choose a rational polyhedral cone decomposition. However, in our case, we have only one choice, namely the torus in a line. We claim that there is a scheme $\mathrm{Sh}_K^{\sim}(GH, Gh)$ such that

- $\mathrm{Sh}_K^{\sim}(GH, Gh)$ is smooth and proper over E.
- $\widetilde{i}_K : \mathrm{Sh}_K(GH, Gh) \hookrightarrow \mathrm{Sh}_K^{\sim}(GH, Gh)$ is an open immersion.
- For $K' \subset K$, there is a morphism $\widetilde{\pi}_K^{K'}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Sh}_{K'}(GH, Gh) \arrow[r, "\widetilde{i}_{K'}"] \arrow[d, "\pi_K^{K'}"'] & \mathrm{Sh}_{K'}^{\sim}(GH, Gh) \arrow[d, "\widetilde{\pi}_K^{K'}"] \\
\mathrm{Sh}_K(GH, Gh) \arrow[r, "\widetilde{i}_K"] & \mathrm{Sh}_K^{\sim}(GH, Gh).
\end{tikzcd}$$

- The boundary $GY_K := \mathrm{Sh}_K^{\sim}(GH, Gh) - \mathrm{Sh}_K(GH, Gh)$ is a smooth divisor defined over E and each geometric component is isomorphic to an extension of an abelian variety of dimension $m-2$ by a finite group.

The boundary part GY_K parameterizes the degeneration of abelian varieties with above PEL data. We consider a semiabelian variety G defined over an algebraically closed field of characteristic 0 with $i : \mathcal{O}_E \hookrightarrow \mathrm{End}(G)$ such that $\mathrm{tr}(i(e); \mathrm{Lie}(G)) = (m-1)e + e^{\tau}$. For every $e \in \mathcal{O}_E$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "t"] & G \arrow[r, "\alpha"] \arrow[d, "i(e)"] & A \arrow[r] & 0 \\
0 \arrow[r] & T \arrow[r, "t"] & G \arrow[r, "\alpha"] & A \arrow[r] & 0.
\end{tikzcd}$$

Then the composition $\alpha \circ i(e) \circ t$ is trivial. Thus i induces actions of $\mathcal{O}_E$ on both torus part T and the abelian variety part A. Suppose the group of cocharacters $X_*(T) \cong \mathbb{Z}^r$ with $r > 0$. Then r is even since E is quadratic imaginary. Further assuming that $\mathrm{tr}(i(e); \mathrm{Lie}(T)) = ae + be^{\tau}$, then $a + b = r$ and $a = b$. Therefore, there is only one possibility, namely $a = b = 1$ and $r = 2$. Then T is of rank 2, and

A is an abelian variety of dimension $m-2$ such that $\mathcal{O}_E$ acts on A and $\mathrm{tr}(i(e);\mathrm{Lie}(A)) = (m-2)e$. Let A_1 be an elliptic curve of CM type $(\mathcal{O}_E, e \mapsto e)$. Then A is isogenous to A_1^{m-2}. Each geometric point s of GY_K corresponds to a semiabelian variety $G_s = (T_s \hookrightarrow G_s \to A_s)$ as above with certain level structure which will be defined later. For two geometric points s, s' in the same geometric connected component, the abelian variety part $A_s \cong A_{s'}$ and the rank 1 $\mathcal{O}_E$-modules $X_*(T_s)$ and $X_*(T_{s'})$ are isomorphic. It is easy to see that if A and T are fixed, then the set of such G, up to isomorphism, is parameterized by $X_*(T) \otimes_{\mathcal{O}_E} A^\vee$ which is an abelian variety of dimension $m-2$.

To include the level structure, we consider only one geometric component since it is same for others. This means that we fix T and A with $\mathcal{O}_E$-actions but, of course, not G. Let us fix a maximal isotropic subspace W of $V_{\mathbb{Z}}$. Then W is of rank 1. We have a filtration $0 \subset W \subset W^\perp \subset V_{\mathbb{Z}}$. Let B_W be the subgroup of $H(\mathbb{A}_{\mathrm{fin}})$ that preserves this filtration, $N_W \subset B_W$ that acts trivially on the associated graded modules, $U_W \subset N_W$ that acts trivially on $W^\perp$ and $V_W = N_W/U_W$. We also fix a generator w of W. On the other hand, we fix an $\mathcal{O}_E \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ generator w_T of $\mathrm{H}_1^{\text{ét}}(T, \widehat{\mathbb{Z}})$ and a polarization $\theta_A : A \to A^\vee$ such that there exists a symplectic similitude of $\mathrm{H}_1^{\text{ét}}(A, \widehat{\mathbb{Z}})$ and $(W^\perp/W) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. For a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_{\mathrm{fin}})$, let $N_{W,K} = N_W \cap K$, $U_{W,K} = U_W \cap K$ and $V_{W,K} = N_{W,K}/U_{W,K}$. Then the level structure of $(G, T \hookrightarrow G \to A)$ with respect to K is a $V_{W,K}$-class of isomorphisms $W^\perp \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \to \mathrm{H}_1^{\text{ét}}(G, \widehat{\mathbb{Z}})$, which sends w to w_T and induces a symplectic similitude of $(W^\perp/W) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\mathrm{H}_1^{\text{ét}}(A, \widehat{\mathbb{Z}}) = \mathrm{H}_1^{\text{ét}}(G, \widehat{\mathbb{Z}})/\mathcal{O}_E \cdot w_T$. We conclude that every geometric component of GY_K is isomorphic to (a connected component of) an extension of $X_*(T) \otimes_{\mathcal{O}_E} A^\vee$ by $V_W/V_{W,H}$ for some T and A as above. There is a universal object $\pi : \mathcal{G} \to \mathrm{Sh}_K^\sim(GH, Gh)$, which is a semiabelian scheme of relative dimension m.

3.2.2 Compactified generating series

We come back to the Shimura variety $\mathrm{Sh}_K := \mathrm{Sh}_K(H, h)$. The above canonical smooth compactification induces a canonical smooth compactification for Sh_K, which we will denote by $\mathrm{Sh}_K^\sim$. Let $Y_K = \mathrm{Sh}_K^\sim - \mathrm{Sh}_K$. They have the same properties as above. We apply the above notations also to the trivial case $m = 2$. Let $\mathscr{L}_K^\sim$ be the line bundle on $\mathrm{Sh}_K^\sim$ induced from $\bigwedge^m \pi_* \Omega_{\mathcal{G}/\mathrm{Sh}_K^\sim(GH,Gh)}$ on $\mathrm{Sh}_K^\sim(GH, Gh)$, which is an extension of the Hodge bundle $\mathscr{L}_K$ on Sh_K. By the canonicality of the compactification, $(\mathscr{L}_K^\sim)_K$ defines an element in $\mathrm{Pic}(\mathrm{Sh}^\sim) := \varinjlim_K \mathrm{Pic}(\mathrm{Sh}_K^\sim)$. We also need to extend special cycles and the generating series. For $1 \leq r < m$ and $x \in V_{\mathrm{fin}}^r := V^r \otimes_{\mathbb{Q}} \mathbb{A}_{\mathrm{fin}}$, we define the *compactified special cycle* to be

$$Z(x)_K^{\sim} = \begin{cases} Z(V_x)_K^{\sim} c_1(\mathscr{L}_K^{\sim,\vee})^{r-\dim_E V_x} & \text{if } V_x \text{ is admissible;} \\ 0 & \text{otherwise,} \end{cases}$$

where $Z(V_x)_K^{\sim}$ is simply the Zariski closure of $Z(V_x)_K$ in $\mathrm{Sh}_K^{\sim}$. We define the *compactified generating series* to be

$$Z_\phi^{\sim}(g) = \begin{cases} \sum_{x \in K \backslash V_{\mathrm{fin}}^r} (\omega_\chi(g)\phi)\,(T(x), x) Z(x)_K^{\sim} & m > 2; \\ \sum_{x \in K \backslash V_{\mathrm{fin}}} (\omega_\chi(g)\phi)\,(T(x), x) Z(x)_K^{\sim} + W_0(\frac{1}{2}, g, \phi) c_1(\mathscr{L}_K^{\sim,\vee}) & r = 1, m = 2, \end{cases}$$

for $g \in H_r(\mathbb{A})$ and $\phi \in \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty}$. It is a formal series in $\mathrm{CH}^r(\mathrm{Sh}^{\sim})_{\mathbb{C}} := \varinjlim_K \mathrm{CH}^r(\mathrm{Sh}_K^{\sim})_{\mathbb{C}}$. Here, $W_0(s, g, \phi) = \prod_v W_0(s, g_v, \phi_v)$ that is holomorphic at $s = \frac{1}{2}$. Moreover, we define the following *positive partial compactified generating series* as

$$Z_\phi^{\sim,+}(g) = \sum_{\substack{x \in K \backslash V_{\mathrm{fin}}^r \\ T(x) \gg \mathbf{0}_r}} (\omega_\chi(g)\phi)\,(T(x), x) Z(x)_K^{\sim},$$

where the sum is taken over all x such that $T(x)$ is totally positive definite. We would like to propose the following conjecture on the modularity of the compactified generating series.

Conjecture 3.2.1. *Let l be a linear functional on $\mathrm{CH}^r(\mathrm{Sh}^{\sim})_{\mathbb{C}}$ such that $l(Z_\phi^{\sim})(g)$ is absolutely convergent. Then*

1. *If $1 \le r \le m-2$, $l(Z_\phi^{\sim})(g)$ is a holomorphic automorphic form on $H_r(\mathbb{A}_F)$.*

2. *If $r = 1, m = 2$, $l(Z_\phi^{\sim})(g)$ is an automorphic form on $H_1(\mathbb{A}_F)$, not necessarily holomorphic.*

3. *In general, if $r = m-1$, $l(Z_\phi^{\sim,+})(g)$ is the sum of the positive definite Fourier coefficients of an automorphic form on $H_{m-1}(\mathbb{A}_F)$.*

The case (2) will be proved in Corollary 3.4.2 and is actually not far from Theorem 3.1.6 as we point out there.

At the end of this section, we discuss the cohomology of the smooth compactification. Fix a rational prime ℓ, there are class maps

$$\mathrm{cl}_K : \mathrm{CH}^r(\mathrm{Sh}_K^{\sim})_{\mathbb{C}} \to \mathrm{H}^{2r}_{\text{ét}}(\mathrm{Sh}_K^{\sim} \times_E E^{\mathrm{ac}}, \mathbb{Z}_\ell(r))^{\Gamma_E} \otimes_{\mathbb{Z}_\ell} \mathbb{C},$$

that are compatible under $\tilde{\pi}_K^{K'}$, where E^{ac} is a fixed algebraic closure of E. They induce

$$\mathrm{cl}:\mathrm{CH}^r(\mathrm{Sh}^\sim)_{\mathbb{C}}\to \mathrm{H}^{2r}_{\text{ét}}(\mathrm{Sh}^\sim\times_E E^{\mathrm{ac}},\mathbb{Z}_\ell(r))^{\Gamma_E}\otimes_{\mathbb{Z}_\ell}\mathbb{C}\subset \mathrm{H}^{2r}_{\mathrm{Bet}}(\mathrm{Sh}^\sim,\mathbb{C}),$$

where

$$\mathrm{H}^{2\bullet}_{\text{ét}}(\mathrm{Sh}^\sim\times_E E^{\mathrm{ac}},\mathbb{Z}_\ell(\bullet))=\varinjlim_K \mathrm{H}^{2\bullet}_{\text{ét}}(\mathrm{Sh}^\sim_K\times_E E^{\mathrm{ac}},\mathbb{Z}_\ell(\bullet));$$
$$\mathrm{H}^{\bullet}_{\mathrm{Bet}}(\mathrm{Sh}^\sim,\mathbb{C})=\varinjlim_K \mathrm{H}^{\bullet}_{\mathrm{Bet}}(\mathrm{Sh}^\sim_K(\mathbb{C}),\mathbb{C}),$$

and the latter one is the Betti cohomology. Let

$$\mathrm{H}^{\bullet}_{Y}(\mathrm{Sh}^\sim,\mathbb{C})=\varinjlim_K \mathrm{H}^{\bullet}_{Y_K}(\mathrm{Sh}^\sim_K(\mathbb{C}),\mathbb{C})$$

be the inductive limit of cohomology groups with support in Y_K as K varies. Then since Y is a smooth divisor, we have

$$\mathrm{H}^{\bullet}_{Y}(\mathrm{Sh}^\sim,\mathbb{C})\cong \mathrm{H}^{\bullet-2}_{\mathrm{Bet}}(Y,\mathbb{C}):=\varinjlim_K \mathrm{H}^{\bullet-2}_{\mathrm{Bet}}(Y_K(\mathbb{C}),\mathbb{C}).$$

Let us denote by $\mathrm{Sh}^\#_K$ the Baily–Borel compactification of Sh_K. Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}^\sim_K \arrow[rr, "j_K"] & & \mathrm{Sh}^\#_K, \\
& \mathrm{Sh}_K \arrow[ul, hook, "\tilde{i}_K"] \arrow[ur, hook', "i^\#_K"'] &
\end{tikzcd}$$

which is compatible and more importantly, Hecke equivariant when K varies. We also denote by $\mathrm{IH}^\bullet(\mathrm{Sh}^\#,\mathbb{C})=\varinjlim_K \mathrm{IH}^\bullet(\mathrm{Sh}^\#_K(\mathbb{C}),\mathbb{C})$ the inductive limit of the intersection cohomology groups, induced by the adjunction maps $\mathrm{IC}^\bullet_{\mathrm{Sh}^\#_K(\mathbb{C})}\to(\pi_K^{K'})_*(\pi_K^{K'})^*\mathrm{IC}^\bullet_{\mathrm{Sh}^\#_{K'}(\mathbb{C})}$ of intersection complexes. Then by [BBD1982, Théorème 6.2.5], we have the following exact sequence

$$\mathrm{H}^{\bullet}_{Y}(\mathrm{Sh}^\sim,\mathbb{C})\longrightarrow \mathrm{H}^{\bullet}_{\mathrm{Bet}}(\mathrm{Sh}^\sim,\mathbb{C})\xrightarrow{j_*}\mathrm{IH}^\bullet(\mathrm{Sh}^\#,\mathbb{C})\longrightarrow 0. \tag{3.8}$$

Let $\mathrm{H}^\bullet_\partial(\mathrm{Sh}^\sim)$ be the image of the first map, which is isomorphic to a quotient of $\mathrm{H}^{\bullet-2}_{\mathrm{Bet}}(Y,\mathbb{C})$.

3.3 Arithmetic theta lifting

We assume Conjecture 3.2.1 and the following assumptions on A-packets, which are a certain part of the Langlands–Arthur conjecture (see [Art1984, Art1989]).

- A-packets are defined for all unitary groups $\mathrm{U}(m)$ defined by a hermitian space over E of rank m. We denote by $\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})$ the set of A-packets of $\mathrm{U}(m)$ and $\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})_{\mathrm{disc}} \subset \mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})$ the subset of discrete A-packets.

- If Π_1 and Π_2 are in $\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})_{\mathrm{disc}}$ such that for almost all $v \in \Sigma$, $\Pi_{1,v}$ and $\Pi_{2,v}$ contain the same unramified representation, then $\Pi_1 = \Pi_2$.

- Let $\mathrm{U}(m)^*$ be the quasi-split unitary group. Then we have the correspondence of A-packets of inner forms: $\mathrm{JL} : \mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})_{\mathrm{disc}} \to \mathrm{AP}(\mathrm{U}(m)^*_{\mathbb{A}_F})_{\mathrm{disc}}$.

Remark 3.3.1. The similar assumptions for orthogonal groups will be proved in the upcoming book of Arthur [Art]. The similar approach should be possible to handle the case of unitary groups as well.

3.3.1 Cohomological triviality

We will fix an incoherent hermitian space $\mathbb{V}$ as above and suppress $\mathbb{H}$ from the notations of Shimura varieties.

Definition 3.3.2 (Arithmetic theta lifting)**.** Let $\pi = \bigotimes'_{v\in\Sigma} \pi_v$ be an irreducible cuspidal automorphic representation of $H_r(\mathbb{A}_F)$ contained in $\mathcal{A}_0(H_r)$: the space of cuspidal automorphic forms on $H_r(\mathbb{A}_F)$. We assume that $1 \leq r \leq m-2$ or $r = 1, m = 2$. For every $\phi \in \mathcal{S}(\mathbb{V}^r)^{\mathrm{U}_\infty}$ and every cusp form $f \in \pi$, the following integral

$$\Theta^f_\phi = \begin{cases} \int_{H_r(F)\backslash H_r(\mathbb{A}_F)} f(g) Z_\phi(g) \mathrm{d}g \in \mathrm{CH}^r(\mathrm{Sh})_{\mathbb{C}} & \mathrm{Sh} \text{ is proper;} \\ \int_{H_r(F)\backslash H_r(\mathbb{A}_F)} f(g) Z^\sim_\phi(g) \mathrm{d}g \in \mathrm{CH}^r(\mathrm{Sh}^\sim)_{\mathbb{C}} & \mathrm{Sh} \text{ is non-proper,} \end{cases}$$

is called the *arithmetic theta lifting* of f, which is a (formal integral of) codimension r cycle on certain (compactified) Shimura variety of dimension $m-1$. The cohomology class of $\Theta^f_\phi \mid \mathrm{Sh}$ is well-defined due to [KM1990].

Remark 3.3.3. The original idea of the arithmetic theta lifting comes from the work of Kudla [Kud2003, Section 8] and Kudla–Rapoport–Yang [KRY2006, Section 9.1]. However, there is an essential difference between our definition and theirs, in the sense that our arithmetic theta lifting is a cocycle on the canonical model of Shimura variety, while theirs (just in the case of Shimura curves) is an Arakelov divisor on certain integral model, which is not canonically defined. It is more natural to define arithmetic theta lifting simply as a cocycle on the generic fiber in order to consider the canonical height.

In the following discussion, let $m = 2n$ and $r = n$. Let π be an irreducible cuspidal automorphic representation of $H_n(\mathbb{A}_F)$, χ a character of $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$, such that π_∞ is a discrete series representation of weight $(n - \frac{\mathfrak{k}^\chi}{2}, n + \frac{\mathfrak{k}^\chi}{2})$ and $\epsilon(\pi, \chi) = -1$. Then the (equal-rank) theta correspondence of π_ι (under ω_χ) is the trivial representation of $\mathrm{U}(2n, 0)_\mathbb{R}$ for every archimedean place ι. Therefore, $\mathbb{V}(\pi, \chi)$ is a totally positive definite incoherent hermitian space over $\mathbb{A}_E$. Now we fix an incoherent hermitian space $\mathbb{V}$ that is totally positive definite of rank $2n$ and let $(\mathrm{Sh}_K)_K$ be the attached (projective system of) Shimura varieties.

We fix an embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ inducing $\iota : F \hookrightarrow \mathbb{C}$ if $F \neq \mathbb{Q}$. Then similarly we have the class map $\mathrm{cl} : \mathrm{CH}^\bullet(\mathrm{Sh}_K)_\mathbb{C} \to \mathrm{H}^{2\bullet}_{\mathrm{Bet}}(\mathrm{Sh}_{K,\iota^\circ}(\mathbb{C}), \mathbb{C})$. By a theorem of S. Zucker [Zuc1982, Section 6] concerning the L^2-cohomology and the intersection cohomology, we have a (compatible system of) Hecke equivariant isomorphisms

$$\mathrm{H}^\bullet_{(2)}(\mathrm{Sh}_K) = \begin{cases} \mathrm{H}^\bullet_{\mathrm{Bet}}(\mathrm{Sh}_{K,\iota^\circ}(\mathbb{C}), \mathbb{C}) & \mathrm{Sh}_K \text{ is proper;} \\ \mathrm{IH}^\bullet(\mathrm{Sh}_K^\#, \mathbb{C}) & \mathrm{Sh}_K \text{ is non-proper.} \end{cases}$$

Let $\mathrm{H}^\bullet_{(2)}(\mathrm{Sh}) = \varinjlim_K \mathrm{H}^\bullet_{(2)}(\mathrm{Sh}_K)$. In the non-proper case, we compose the map j_* in (3.8) to get a class map, which is still denoted by $\mathrm{cl} : \mathrm{CH}^\bullet(\mathrm{Sh}^\sim) \to \mathrm{H}^{2\bullet}_{(2)}(\mathrm{Sh})$. We have the following proposition.

Proposition 3.3.4. *The class* $\mathrm{cl}(\Theta^f_\phi) = 0$ *in* $\mathrm{H}^{2n}_{(2)}(\mathrm{Sh})$*, i.e., if* Sh *is proper (resp. non-proper),* Θ^f_ϕ *is cohomologically trivial (resp. such that* $\mathrm{cl}(\Theta^f_\phi) \in \mathrm{H}^{2n}_\partial(\mathrm{Sh}^\sim)$*).*

Proof. If Sh is non-proper, we can assume that $n > 1$. By our definition of the arithmetic theta lifting, for $\phi = \phi_\infty \otimes \phi_{\mathrm{fin}}$ with fixed ϕ_∞, $\mathrm{cl}(\Theta^f_\phi)$ defines an element in

$$\mathrm{Hom}_{H_n(\mathbb{A}_{F,\mathrm{fin}})}\left(\mathcal{S}(\mathbb{V}^n_{\mathrm{fin}}) \otimes \pi_{\mathrm{fin}}, \mathrm{H}^{2n}_{(2)}(\mathrm{Sh})\right),$$

where $H_n(\mathbb{A}_{F,\mathrm{fin}})$ acts trivially on the L^2-cohomology.

Let $V^{(\iota)}$ be the nearby hermitian space of $\mathbb{V}$ at ι (*cf.* 3.1.1) and $H^{(\iota)} = \mathrm{U}(V^\iota)$. Then since $Z^\sim_{\omega_\chi(h)\phi}(g) = \mathrm{T}^*_h Z^\sim_\phi(g)$ for all $h \in H^{(\iota)}(\mathbb{A}_{F,\mathrm{fin}})$, where T_h is the Hecke operator of h, we see that

$\mathrm{cl}(\Theta_\phi^f)$ in fact defines an element

$$\begin{aligned} H_{\Theta,\phi_\infty} &\in \mathrm{Hom}_{H_n(\mathbb{A}_{F,\mathrm{fin}})\times H^{(\iota)}(\mathbb{A}_{F,\mathrm{fin}})}\left(\mathcal{S}(\mathbb{V}_{\mathrm{fin}}^n)\otimes\pi_{\mathrm{fin}}, \mathrm{H}^{2n}_{(2)}(\mathrm{Sh})\right) \\ &= \mathrm{Hom}_{H_n(\mathbb{A}_{F,\mathrm{fin}})\times H^{(\iota)}(\mathbb{A}_{F,\mathrm{fin}})}\left(\mathcal{S}(\mathbb{V}_{\mathrm{fin}}^n), \pi_{\mathrm{fin}}^\vee\otimes \mathrm{H}^{2n}_{(2)}(\mathrm{Sh})\right), \end{aligned}$$

where $H_n(\mathbb{A}_{F,\mathrm{fin}})\times H^{(\iota)}(\mathbb{A}_{F,\mathrm{fin}})$ acts on $\mathcal{S}(\mathbb{V}_{\mathrm{fin}}^n)$ through the Weil representation ω_χ, and $H^{(\iota)}(\mathbb{A}_{F,\mathrm{fin}})$ acts on $\mathrm{H}^{2\bullet}_{(2)}(\mathrm{Sh})$ through Hecke operators and on π_{fin} trivially. As an $H^{(\iota)}(\mathbb{A}_{F,\mathrm{fin}})$-representation, we have the following well-known decomposition (*cf.* e.g., [BW2000, Chapter XIV])

$$\mathrm{H}^{2n}_{(2)}(\mathrm{Sh}) = \bigoplus_\sigma m_{\mathrm{disc}}(\sigma)\mathrm{H}^{2n}(\mathrm{Lie}\, H_\infty^{(\iota)}, \mathcal{K}_{H_\infty^{(\iota)}}; \sigma_\infty)\otimes\sigma_{\mathrm{fin}},$$

where the direct sum is taken over all irreducible discrete automorphic representations of $H^{(\iota)}(\mathbb{A}_F)$. If the invariant functional $H_{\Theta,\phi_\infty}\neq 0$, then some σ_{fin} with

$$m_{\mathrm{disc}}(\sigma_\infty\otimes\sigma_{\mathrm{fin}})\mathrm{H}^{2n}(\mathrm{Lie}\, H_\infty^{(\iota)}, \mathcal{K}_{H_\infty^{(\iota)}}; \sigma_\infty)\neq 0$$

is the theta correspondence $\theta(\pi_{\mathrm{fin}}^\vee)$ of $\pi_{\mathrm{fin}}^\vee$.

We define a character $\tilde{\chi}$ of $E^{\times,1}\backslash\mathbb{A}_E^{\times,1}$ in the following way. For every $x\in\mathbb{A}_E^{\times,1}$, we can write $x=\frac{e}{e^\tau}$ for some $e\in\mathbb{A}_E^\times$ and define $\tilde{\chi}(x)=\chi(e)$ that is well-defined since $\chi|_{\mathbb{A}_F^\times}=1$.

For all finite place v such that $v\nmid 2$ and ψ_v, χ_v, π_v unramified, we have $H_v^{(\iota)}\cong H_{n,v}$. Let $\Sigma\in\mathrm{AP}(H^{(\iota)}_{\mathbb{A}_F})_{\mathrm{disc}}$ be the A-packet containing σ and $\Pi\in\mathrm{AP}(H_{n,\mathbb{A}_F})_{\mathrm{disc}}$ containing π. Then by Corollary A.3.6, we have that for v as above, $\mathrm{JL}(\Sigma)_v=\mathrm{JL}(\Sigma_v)=\Sigma_v$ and $\Pi_v\otimes\tilde{\chi}_v$ contain the same unramified representation. Therefore, $\mathrm{JL}(\Sigma)$ and $\Pi\otimes\tilde{\chi}$ coincide. In particular,

$$\mathrm{JL}(\Sigma_\infty)=\mathrm{JL}(\Sigma)_\infty=\Pi_\infty\otimes\tilde{\chi}_\infty,$$

which implies that Σ_∞ is a discrete series L-packet (*cf.* [Ada2011]). This contradicts our assumption since for every discrete series representation σ_∞, $\mathrm{H}^\bullet(\mathrm{Lie}\, H_\infty^{(\iota)}, \mathcal{K}_{H_\infty^{(\iota)}}; \sigma_\infty)\neq 0$ happens only in the middle dimension, which is $2n-1$, not $2n$ (*cf.* [BW2000, Chapter II, Theorem 5.4]). Thus $H_{\Theta,\phi_\infty}=0$ and we prove the proposition. □

The proposition says that Θ_ϕ^f is automatically cohomologically trivial at least in the proper case. We would like to propose the following conjecture.

Conjecture 3.3.5. *When* Sh *is non-proper,* $\mathrm{cl}(\Theta_\phi^f)\in\mathrm{H}^{2n}_\partial(\mathrm{Sh}^\sim)$ *is* 0 *for every cusp form* $f\in\pi$ *and*

ϕ as above.

When $n=1$, this follows from Corollary 3.4.2 simply by computing the degree of the generating series, which is the linear combination of an Eisenstein series and (possibly) an automorphic character (*i.e.* 1-dimensional automorphic representation) of $H_1(\mathbb{A}_F)$. Therefore, $\mathrm{cl}(\Theta_\phi^f)$ is zero since f is cuspidal. For the general case, we believe that the same phenomenon will happen.

3.3.2 Arithmetic inner product formula: the general conjecture

Let us further assume Conjecture 3.3.5 and the existence of *Beilinson–Bloch height pairing* [Beĭ1987, Blo1984] on smooth proper schemes over number fields. Therefore, we may let

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}}^K$$

be the Beilinson–Bloch height pairing on Sh_K (resp. $\mathrm{Sh}_K^\sim$) if it is proper (resp. non-proper) for sufficiently small K. Let $\mathrm{Vol}(K)$ be the volume defined in Definition 4.3.3. Then

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} := \mathrm{Vol}(K) \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}}^K$$

is a well-defined number that is independent of K.

If $\mathbb{V} \not\cong \mathbb{V}(\pi,\chi)$, then $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} = 0$ for every f, $f^\vee$ and ϕ, $\phi^\vee$ since otherwise, it defines a nonzero functional

$$\gamma(f, f^\vee, \phi, \phi^\vee) \in \mathrm{Hom}_{H_n(\mathbb{A}_{F,\mathrm{fin}}) \times H_n(\mathbb{A}_{F,\mathrm{fin}})} \left(R(\mathbb{V}_{\mathrm{fin}}, \chi_{\mathrm{fin}}), \pi_{\mathrm{fin}}^\vee \boxtimes \chi_{\mathrm{fin}} \pi_{\mathrm{fin}} \right),$$

which contradicts the fact that the latter space is zero. This will imply that, assuming the conjecture that the Beilinson–Bloch height pairing is non-degenerate, every arithmetic theta lifting $\Theta_\phi^f = 0$.

If $\mathbb{V} \cong \mathbb{V}(\pi)$, then we have the following main conjecture.

Conjecture 3.3.6 (Arithmetic inner product formula)**.** *Let π be an irreducible cuspidal automorphic representation of $H_n(\mathbb{A}_F)$, χ a character of $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$, such that π_∞ is a discrete series representation of weight $(n - \frac{\mathfrak{k}_\chi}{2}, n + \frac{\mathfrak{k}_\chi}{2})$, $\epsilon(\pi,\chi) = -1$ and $\mathbb{V} \cong \mathbb{V}(\pi,\chi)$. Then for every $f \in \pi$, $f^\vee \in \pi^\vee$ and every $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V}^n)^{\mathrm{U}_\infty}$ that are decomposable, we have*

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$

where in the last product, almost all factors are 1.

We would like to remark that when $n=1$, the height pairing $\langle \Theta^{f}_{\phi}, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{BB}}$ is simply the classical *Néron–Tate height pairing*, which will be recalled in 3.4.1, hence is defined *unconditionally*.

3.4 Arithmetic kernel functions

3.4.1 Néron–Tate height pairing on curves

We will review the general theory of the Néron–Tate height pairing on curves defined over number fields and some related facts.

Let E be a number field, not necessarily CM; M a connected smooth projective curve over E, not necessarily geometrically connected. Let $\mathrm{CH}^1(M)^0_{\mathbb{C}}$ be the group of *cohomologically trivial cycles*, which is the kernel of the following map

$$\deg : \mathrm{CH}^1(M)_{\mathbb{C}} \to \mathrm{H}^2_{\text{ét}}(M_{E^{\mathrm{ac}}}, \mathbb{Z}_\ell(1))^{\Gamma_E} \otimes_{\mathbb{Z}_\ell} \mathbb{C} \cong \mathbb{C}$$

for a fixed rational prime number ℓ. Let $\mathcal{M}$ be a regular model of M, *i.e.* a regular scheme, flat and projective over $\operatorname{Spec} \mathcal{O}_E$ with the generic fibre $\mathcal{M}_E \cong M$.

An *arithmetic divisor* is a datum $(\mathcal{Z}, g_{\iota^\circ})$, where $\mathcal{Z} \in \mathrm{Z}^1(\mathcal{M})_{\mathbb{C}}$ is a usual divisor and g_{ι° is a Green function, *i.e.* Green (0,0)-form of logarithmic type [Sou1992, II.2], for the divisor $\mathcal{Z}_{\iota^\circ}(\mathbb{C})$ on $\mathcal{M}_{\iota^\circ}(\mathbb{C})$ for each embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$. We denote by $\widehat{\mathrm{Z}}^1_{\mathbb{C}}(\mathcal{M})$ the group of arithmetic $\mathbb{C}$-divisors. For a nonzero rational function f on $\mathcal{M}$, we define the associated *principal arithmetic divisor* to be

$$\widehat{\operatorname{div}}(f) = \left(\operatorname{div}(f), -\log |f_{\iota^\circ,\mathbb{C}}|^2\right).$$

The quotient of $\widehat{\mathrm{Z}}^1_{\mathbb{C}}(\mathcal{M})$ divided by the $\mathbb{C}$-subspace generated by principal arithmetic divisors is the *arithmetic Chow group*, denoted by $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$. Inside $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$, there is a subspace $\mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})_{\mathbb{C}}$, which is generated by $(\mathcal{Z}, 0)$ with $\mathcal{Z}$ supported on special fibres. Let $\mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})^{\perp}_{\mathbb{C}} \subset \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$ be the orthogonal complement under the $\mathbb{C}$-bilinear Gillet-Soulé height pairing (or Arakelov height pairing)

$$\langle -, - \rangle_{\mathrm{GS}} : \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M}) \times \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M}) \to \mathbb{C}.$$

An arithmetic divisor $(\mathcal{Z}, g_{\iota^\circ})$ is *flat* if we have the following equality in the space $\mathrm{D}^{1,1}(M_{\iota^\circ}(\mathbb{C}))$ of

$(1,1)$-currents on $M_{\iota^\circ}(\mathbb{C})$:

$$\mathrm{dd}^c[g_{\iota^\circ}] + \delta_{\mathcal{Z}_{\iota^\circ}(\mathbb{C})} = 0$$

for every ι°, where $\mathrm{d}^c = (4\pi i)^{-1}(\partial - \overline{\partial})$; $[-]$ is the associated current; and δ is the Dirac current. Flatness is well-defined in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$. Now we introduce the subgroup $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})^0$ of $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$ consisting of elements (represented by) $(\mathcal{Z}, g_{\iota^\circ})$ satisfying:

1. $(\mathcal{Z}, g_{\iota^\circ})$ is inside $\mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})^{\perp}_{\mathbb{C}}$.

2. $\mathcal{Z}_E$ is inside $\mathrm{CH}^1(M)^0_{\mathbb{C}}$.

3. $(\mathcal{Z}, g_\iota)$ is flat.

Therefore, we have a natural map

$$\begin{aligned} p_{\mathcal{M}} : \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})^0 &\to \mathrm{CH}^1(M)^0_{\mathbb{C}} \\ (\mathcal{Z}, g_\iota) &\mapsto \mathcal{Z}_E, \end{aligned} \tag{3.9}$$

which is surjective. We define the *Néron–Tate height pairing* as

$$\begin{aligned} \langle -, - \rangle_{\mathrm{NT}} : \mathrm{CH}^1(M)^0_{\mathbb{C}} \times \mathrm{CH}^1(M)^0_{\mathbb{C}} &\to \mathbb{C} \\ (Z_1, Z_2) &\mapsto \langle (\mathcal{Z}_1, g_{1,\iota^\circ}), (\mathcal{Z}_2, g_{2,\iota^\circ}) \rangle_{\mathrm{GS}}, \end{aligned} \tag{3.10}$$

where $(\mathcal{Z}_i, g_{i,\iota^\circ})$ $(i = 1, 2)$ is any preimage of Z_i under $p_{\mathcal{M}}$. It is easy to see that the resulting pairing is independent of the choices of preimages and also the regular model $\mathcal{M}$.

In practice, the cycles we are interested in are not automatically cohomologically trivial. We need to make some modifications with respect to some auxiliary data. This is quite easy if we are working over a curve. Let $\widehat{\mathrm{Pic}}(M)$ be the abelian group of isomorphism classes of *hermitian line bundles* on M. Recall that a hermitian line bundle is the data $\overline{\mathscr{L}} = (\mathscr{L}, \|\bullet\|_{\iota^\circ})$, where $\mathscr{L} \in \mathrm{Pic}(M)$ and $\|\bullet\|_{\iota^\circ}$ is a (smooth) hermitian metric on the holomorphic line bundle $\mathscr{L}_{\iota^\circ,\mathbb{C}}$. We assume that $\deg c_1(\mathscr{L}) \neq 0$. For every $Z \in \mathrm{CH}^1(M)$, the divisor

$$Z^0_{\mathscr{L}} = Z - \frac{\deg Z}{\deg c_1(\mathscr{L})} c_1(\mathscr{L})$$

is in $\mathrm{CH}^1(M)^0_{\mathbb{C}}$.

Now we define the modified height pairing with respect to $\overline{\mathscr{L}}$ to be

$$\langle Z_1, Z_2 \rangle_{\overline{\mathscr{L}}} := \langle Z^0_{1,\mathscr{L}}, Z^0_{2,\mathscr{L}} \rangle_{\mathrm{NT}}$$

for every $Z_i \in \mathrm{CH}^1(M)_{\mathbb{C}}$ $(i = 1, 2)$. In particular, we need to choose suitable Green function on Z_i when computing via (3.10). We say that the Green function g_{ι° of Z is $\overline{\mathscr{L}}$*-admissible* if the following equalities of $(1,1)$-currents hold

$$\mathrm{dd}^c[g_{\iota^\circ}] + \delta_{Z_{\iota^\circ}(\mathbb{C})} = \frac{\deg Z}{\deg c_1(\mathscr{L})}[c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \| \bullet \|_{\iota^\circ})];$$
$$\int_{M_{\iota^\circ}(\mathbb{C})} g_{\iota^\circ} \cdot c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \| \bullet \|_{\iota^\circ}) = 0,$$

where $c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \| \bullet \|_{\iota^\circ}) \in \mathrm{A}^{1,1}(M_{\iota^\circ}(\mathbb{C}))$ is the *Chern form* associated to the hermitian holomorphic line bundle $(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \| \bullet \|_{\iota^\circ})$, which is a $(1,1)$-form.

3.4.2 Degree of generating series

We adopt the construction in 3.1.1 in the case where $m = 2$ and $r = 1$. In particular, $\mathbb{V}$ is a totally positive definite hermitian space over $\mathbb{A}_E$ of rank 2, and we write $H' = H_1$. Let $\mathbb{H} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\,\mathrm{U}(\mathbb{V})$ be the associated unitary group over $\mathbb{A}$. For every (sufficiently small; *cf.* Assumption 3.1.1) open compact subgroup K of $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})$, there is a Shimura *curve* $\mathrm{Sh}_K(\mathbb{H})$, smooth over the reflex field E. For every embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ over $\iota \in \Sigma_\infty$, we have the following ι°-adic uniformization

$$\mathrm{Sh}_K(\mathbb{H})^{\mathrm{an}}_{\iota^\circ} \cong H^{(\iota)}(\mathbb{Q}) \backslash \left(\mathcal{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_{\mathrm{fin}})/K \right).$$

The underlying real symmetric domain $\mathcal{D}^{(\iota^\circ)}$ is identified with the $H^\iota(\mathbb{R})$-conjugacy class of the Hodge map $h^{(\iota)} : \mathbb{S} \to H^{(\iota)}_{\mathbb{R}} \cong \mathrm{U}(1,1)_{\mathbb{R}} \times \mathrm{U}(2,0)^{d-1}_{\mathbb{R}}$ given by

$$h^{(\iota)}(z) = \left(\begin{pmatrix} 1 & \\ & \overline{z}/z \end{pmatrix}, \mathbf{1}_2, \dots, \mathbf{1}_2 \right).$$

The Shimura curves $\mathrm{Sh}_K(\mathbb{H})$ are non-proper if and only if $F = \mathbb{Q}$ and $\Sigma(\mathbb{V}) = \{\infty\}$. In this case, we can compactify them by adding cusps. We denote by M_K the compactified (resp. original) Shimura curve if $\mathrm{Sh}_K(\mathbb{H})$ is non-proper (resp. proper), and M the projective system $(M_K)_K$ with respect to the projections $\pi^{K'}_K : M_{K'} \to M_K$ (3.1). On each M_K, we have the Hodge bundle $\mathscr{L}_K \in \mathrm{Pic}(M_K)_{\mathbb{Q}}$ that is ample. They are compatible under pullbacks of $\pi^{K'}_K$, hence define an element $\mathscr{L} \in \mathrm{Pic}(M)_{\mathbb{Q}} :=$

$\varinjlim_K \mathrm{Pic}(M_K)_{\mathbb{Q}}$.

Adapting the definition of (compactified) generating series in 3.2.2, we have

$$Z_\phi(g) = \sum_{x \in K \backslash \mathbb{V}_{\mathrm{fin}}} \omega_\chi(g)\phi(T(x), x) Z(x)_K,$$

$$Z^\sim_\phi(g) = \begin{cases} Z_\phi(g) & \text{if } \mathrm{Sh}(\mathbb{H})_K \text{ is proper;} \\ Z_\phi(g) + W_0(\frac{1}{2}, g, \phi) c_1(\mathscr{L}_K^\vee) & \text{if not,} \end{cases}$$

respectively, as vector-valued series in $\mathrm{CH}^1(M_K)_{\mathbb{C}}$ for $\phi \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty K}$ and $g \in H'(\mathbb{A}_F)$, where $W_0(s, g, \phi) = \prod_v W_0(s, g_v, \phi_v)$ that is holomorphic at $s = \frac{1}{2}$. It is easy to see that $Z_\phi(g)$ and $Z^\sim_\phi(g)$ are compatible under pullbacks of $\pi_K^{K'}$, hence define series in $\mathrm{CH}^1(M)_{\mathbb{C}} := \varinjlim_K \mathrm{CH}^1(M_K)_{\mathbb{C}}$. Readers may view the modification in the non-proper case as an analogy of the classical Eisenstein series $G_2(\tau)$ (which is not a modular form!). It becomes modular if we add a term $-\pi/\operatorname{Im}\tau$ at the price of being non-holomorphic (*cf.* e.g., [DS2005, Page 18]).

We apply the construction in 3.4.1 to the curve M_K. The cycles whose heights we want to compute are the generating series $Z^\sim_\phi(g)$, which are not necessarily cohomologically trivial. We use the dual Hodge bundle $\mathscr{L}^\vee = (\mathscr{L}_K^\vee)_K \in \mathrm{Pic}(M)$ to modify as in 3.4.1. The metric on $\mathscr{L}_{\iota^\circ, \mathbb{C}}$ for some $\iota^\circ \in \Sigma_\infty^\circ$ over $\iota \in \Sigma_\infty$ is the one descended from the H'_ι-invariant metric

$$\|v\|_{\iota^\circ} = \frac{1}{2}(v, v)_\iota$$

for $v \in V_\iota^{(\iota)}$ and the hermitian form $(-,-)_\iota$ of $V^{(\iota)}$ at ι. We denote by $\overline{\mathscr{L}} = (\overline{\mathscr{L}}_K)_K \in \widehat{\mathrm{Pic}}(M)$ the corresponding metrized line bundle. Since $\mathscr{L}$ is ample, $\deg c_1(\mathscr{L}_K) \neq 0$. For $\phi \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty K}$ and $g \in H'(\mathbb{A}_F)$, we define the *arithmetic theta series* to be

$$\Theta_\phi(g) = Z^\sim_\phi(g) - \frac{\deg Z^\sim_\phi(g)}{\deg c_1(\mathscr{L}_{K'}^\vee)} c_1(\mathscr{L}_{K'}^\vee)$$

on every curve $M_{K'}$ with $K' \subset K$. The ratio

$$D(g, \phi) := \frac{\deg Z^\sim_\phi(g)}{\deg c_1(\mathscr{L}_{K'}^\vee)}$$

is independent of the choice of K'.

We compute the degree function $D(g, \phi)$. From $\phi \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty K}$ that is decomposable, we can form an Eisenstein series

$$E(s,g,\phi) = \sum_{\gamma \in P'(F)\backslash H'(F)} \left(\omega_\chi(\gamma g)\phi\right)(0)\lambda_{P'}(\gamma g)^{s-\frac{1}{2}}$$

on $H'(\mathbb{A}_F)$, which is absolutely convergent if $\Re(s) > \frac{1}{2}$ and has a meromorphic continuation to the entire complex plane. We take Tamagawa measures (with respect to ψ) $\mathrm{d}h$ on $\mathbb{H}(\mathbb{A})$, $\mathrm{d}\bar{h}$ on $\mathbb{A}_E^{\times,1} = \mathbb{H}/\mathbb{H}^{\mathrm{der}}(\mathbb{A})$ and $\mathrm{d}h_x$ on $\mathbb{H}(\mathbb{A})_x$, which is the stabilizer of $x \in \mathbb{V}$ in $\mathbb{H}(\mathbb{A})$.

For every $v \in \Sigma$, let $b \in F_v^\times$ such that $\Omega_b := \{x \in \mathbb{V}_v \mid T(x) = b\} \neq \emptyset$. Then the local Whittaker integral $W_b(s, e, \phi_v)$ has a holomorphic continuation to the entire complex plane and $W_b(\frac{1}{2}, e, \phi_v)$ is not identically zero. Therefore, we have an N_v-intertwining map

$$\begin{aligned} \mathcal{S}(\mathbb{V}_v) &\to \mathbb{C}_{N_v,\phi_b} \\ \phi_v &\mapsto W_b(\frac{1}{2}, e, \phi_v). \end{aligned}$$

On the other hand, by [Ral1987, Lemma 4.2] for v finite, and [Ral1987, Lemma 4.2] and [KR1994, Proposition 2.10] for v infinite (see also [Ich2004, Proposition 6.2]), we have

$$W_b(\frac{1}{2}, e, \phi_v) = \gamma_{\mathbb{V}_v} \int_{\Omega_b} \phi_v(x)\mathrm{d}\mu_{v,b}(x) \tag{3.11}$$

for the quotient measure $\mathrm{d}\mu_{v,b} = \mathrm{d}h_v/\mathrm{d}h_{v,x}$ on Ω_b for every $x \in \Omega_b$.

Proposition 3.4.1. *The Eisenstein series $E(s,g,\phi)$ is holomorphic at $s = \frac{1}{2}$ and*

$$D(g,\phi) = E(s,g,\phi)\mid_{s=\frac{1}{2}}.$$

Proof. We can assume that ϕ is decomposable. For $b \in F^\times$, let

$$D_b(g,\phi) = \frac{1}{\deg c_1(\mathscr{L}_{K'}^\vee)} \sum_{\substack{x \in K'\backslash \mathbb{V}_{\mathrm{fin}} \\ T(x)=b}} \left(\omega_\chi(g)\phi\right)(b,x) \deg Z(x)_{K'}$$

be the b-th Fourier coefficient of $D(g,\phi)$. We first compute the degree of $Z(x)_{K'}$ when $T(x) = b$ is totally positive. Without lost of generality, let us assume that x is contained in the image of some (rational) nearby hermitian space $V^{(\iota)} \hookrightarrow \mathbb{V}_{\mathrm{fin}}$ and K' is sufficiently small. The isomorphism

$\det : H_x^{(\iota)} \to E^{\times,1}$ induces a surjective map

$$H_x^{(\iota)} \backslash \mathbb{H}(\mathbb{A}_{\mathrm{fin}})_x / (K' \cap \mathbb{H}(\mathbb{A}_{\mathrm{fin}})_x) \to E^{\times,1} \backslash \mathbb{A}_{\mathrm{fin},E}^{\times,1} / \det K'.$$

Therefore,

$$\deg Z(x)_{K'} = \left| \frac{\det K'}{K' \cap \mathbb{H}(\mathbb{A}_{\mathrm{fin}})_x} \right| = \frac{\mathrm{Vol}(\det K', \mathrm{d}\bar{h}_{\mathrm{fin}})}{\mathrm{Vol}(K' \cap \mathbb{H}(\mathbb{A}_{\mathrm{fin}})_x, \mathrm{d}h_{f,x})}.$$

When $b \neq 0$ and is not totally positive, $\deg Z(x)_{K'} = 0$ by definition. Therefore,

$$\begin{aligned} D_b(g,\phi) &= \frac{1}{\deg c_1(\mathscr{L}_{K'}^\vee)} \sum_{\substack{x \in K' \backslash \mathbb{V}_{\mathrm{fin}} \\ T(x)=b}} (\omega_\chi(g)\phi)(b,x) \frac{\mathrm{Vol}(\det K')}{\mathrm{Vol}(K' \cap \mathbb{H}(\mathbb{A}_{\mathrm{fin}})_x)} \\ &= \frac{\omega_\chi(g_\infty)\phi_\infty(b)\,\mathrm{Vol}(\det K')}{\deg c_1(\mathscr{L}_{K'}^\vee)\,\mathrm{Vol}(K')} \int_{\substack{x \in \mathbb{V}_{\mathrm{fin}} \\ T(x)=b}} (\omega_\chi(g)\phi)(x) \mathrm{d}\mu_b(x) \\ &= \frac{\omega_\chi(g_\infty)\phi_\infty(b)\,\mathrm{Vol}(\det K')}{\deg c_1(\mathscr{L}_{K'}^\vee)\,\mathrm{Vol}(K')} \prod_{v \in \Sigma_{\mathrm{fin}}} \int_{\Omega_b} (\omega_\chi(g_v)\phi_v)(x) \mathrm{d}\mu_{v,b}(x) \end{aligned}$$

for b totally positive, and $D_b(g,\phi) = 0$ otherwise.

On the other hand, $E_b(s,g,\phi)$ is holomorphic at $s = \frac{1}{2}$ for $b \neq 0$. For b not totally positive, $E_b(s,g,\phi)|_{s=\frac{1}{2}} = 0$; otherwise,

$$\begin{aligned} E_b(s,g,\phi)|_{s=\frac{1}{2}} &= W_b(\frac{1}{2},g,\phi) = \prod_{v \in \Sigma} W_b(\frac{1}{2},g_v,\phi_v) \\ &\overset{(3.11)}{=} \prod_{v \in \Sigma} \gamma_{\mathbb{V}_v} \int_{\Omega_b} \omega_\chi(g_v)\phi_v(x) \mathrm{d}\mu_{v,b}(x) \\ &= -\mathrm{Vol}(\Omega_\infty)\,(\omega_\chi(g_\infty)\phi_\infty)(b) \prod_{v \in \Sigma_{\mathrm{fin}}} \int_{\Omega_b} (\omega_\chi(g_v)\phi_v)(x) \mathrm{d}\mu_{v,b}(x), \end{aligned}$$

where $\mathrm{Vol}(\Omega_\infty) = \mathrm{Vol}(\Omega_{\infty,b})$ for every b that is totally positive. Let

$$D = \frac{\mathrm{Vol}(\det K')}{\mathrm{Vol}(\Omega_\infty) \deg c_1(\mathscr{L}_{K'})\,\mathrm{Vol}(K')}.$$

Now we compute the constant term

$$D_0(g,\phi) = \omega_\chi(g)\phi(0) + W_0(\frac{1}{2},g,\phi),$$

and the constant term of $E(\frac{1}{2}, g, \phi)$ is

$$E_0(\frac{1}{2}, g, \phi) = \omega_\chi(g)\phi(0) + W_0(\frac{1}{2}, g, \phi).$$

Here the intertwining term $W_0(\frac{1}{2}, g, \phi)$ is nonzero only if $\mathrm{Sh}_K(\mathbb{H})$ is not proper, *i.e.* $|\Sigma(\mathbb{V})| = 1$. There are two cases:

- If $\mathrm{Sh}_K(\mathbb{H})$ is proper, then we can apply Theorem 3.1.6 to see that $D(g, \phi)$ is already an automorphic form. Comparing the ratio of the constant term and non-constant terms, we find that $D = 1$.

- If $\mathrm{Sh}_K(\mathbb{H})$ is not proper, we calculate the degree of the Hodge bundle via the classical way on modular curves and find that $D = 1$.

Therefore, the proposition follows. □

We let

$$E(g, \phi) = E(s, g, \phi)|_{s=\frac{1}{2}} - W_0(\frac{1}{2}, g, \phi).$$

Then

$$\Theta_\phi(g) = Z_\phi(g) - E(g, \phi)c_1(\mathscr{L}_K^\vee).$$

If $|\Sigma(\mathbb{V})| > 1$, $W_0(\frac{1}{2}, g, \phi) = 0$; otherwise, it equals $C\,(\tilde{\chi} \circ \det)$ where C is a constant and $\tilde{\chi}$ is the descent of χ to $\mathbb{A}_E^{\times,1}$, as in the proof of Proposition 3.3.4. In all cases, $E(g, \phi)$ is a linear combination of an Eisenstein series and an automorphic character.

Proposition 3.4.1 implies the following corollary on the modularity of the generating series in the compactified case.

Corollary 3.4.2. *Let l be a linear functional of $\mathrm{CH}^1(M)_\mathbb{C}$. Then $l(Z_\phi^\sim)(g)$ and hence $l(\Theta_\phi)(g)$ are absolutely convergent and are automorphic forms of H'.*

Proof. Assume that ϕ is invariant under $K \subseteq \mathbb{H}(\mathbb{A}_{\mathrm{fin}})$. We only need to prove that the identity $Z_\phi^\sim(\gamma g) = Z_\phi^\sim(g)$ in $\mathrm{Pic}(M_K)_\mathbb{C}$ for every $\gamma \in H'(\mathbb{Q})$. By Theorem 3.1.6 and the fact that the Hodge bundle is supported on the cusps, $Z_\phi^\sim(\gamma g) = Z_\phi^\sim(g)$ in $\mathrm{CH}^1(\mathrm{Sh}_K(\mathbb{H}))_\mathbb{C}$. So their difference must be supported on the set of cusps. By a theorem of Manin–Drinfeld (*cf.* [Man1972, Dri1973]) which posits that every two cusps are same in $\mathrm{CH}^1(M_K)_\mathbb{C}$, we have an exact sequence

$$\mathbb{C} \to \mathrm{CH}^1(M_K)_\mathbb{C} \to \mathrm{CH}^1(\mathrm{Sh}_K(\mathbb{H}))_\mathbb{C} \to 0.$$

Therefore, we only need to prove that $\deg Z^{\sim}_{\phi}(\gamma g) = \deg Z^{\sim}_{\phi}(g)$, which is true by the Proposition 3.4.1. □

In particular, Definition 3.3.2 specializes to the following unconditional definition.

Definition 3.4.3. Let π be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F) = H_1(\mathbb{A}_F)$. For every cusp form $f \in \pi$ and $\phi \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty}$, we define the *arithmetic theta lifting* of f to be the following integral

$$\Theta^f_\phi = \int_{H'(F)\backslash H'(\mathbb{A}_F)} f(g)\Theta_\phi(g)\mathrm{d}g \in \mathrm{CH}^1(M)^0_{\mathbb{C}},$$

which is a divisor on the projective system of (compactified) Shimura curves.

3.4.3 Decomposition of arithmetic kernel functions

For $\Phi = \sum_{i=1}^s \phi_{i,1} \otimes \phi_{i,2}$ with $\phi_{i,\alpha} \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty\, K}$ for $\alpha = 1, 2$, we define the *geometric kernel function* associated to the test function Φ to be

$$\mathbb{E}(g_1, g_2; \Phi) := \mathrm{Vol}(K') \sum_{i=1}^{s} \langle \Theta_{\phi_{i,1}}(g_1), \Theta_{\phi_{i,2}}(g_2)\rangle^{K'}_{\mathrm{NT}},$$

where the superscript K' means that we are taking the Néron–Tate height pairing on the curve $M_{K'}$ for some $K' \subset K$, of which the definition is independent. By Corollary 3.4.2, $\mathbb{E}(g_1, g_2; \Phi)$ is in $\mathcal{A}(H' \times H')$. Now let us work over M_K and choose a regular model $\mathcal{M}_K$ of it. We choose an arithmetic line bundle $\widehat{\omega}_K$ extending $\overline{\mathscr{L}}^{\vee}_K$. In particular, the metrics on $\widehat{\omega}_K$ at archimedean places are same as those on $\overline{\mathscr{L}}^{\vee}_K$.

Since the map $p_{\mathcal{M}_K}$ (3.9) is surjective, we may fix an inverse linear map $p^{-1}_{\mathcal{M}_K}$ and write

$$\widehat{\Theta}_\phi(g) := p^{-1}_{\mathcal{M}_K}(\Theta_\phi(g)) = ([Z_\phi(g)]^{\mathrm{Zar}}, g_{\iota^\circ}) + (\mathcal{V}_\phi(g), 0) - E(g, \phi)\widehat{\omega}_K,$$

where g_{ι° is an $\overline{\mathscr{L}}_K$-admissible Green function of $Z_\phi(g)$, and $\mathcal{V}_\phi(g)$ is the sum of (finitely many) vertical components supported on special fibres. We also simply write

$$\widehat{Z}_\phi(g) = ([Z_\phi(g)]^{\mathrm{Zar}}, g_{\iota^\circ}) + (\mathcal{V}_\phi(g), 0).$$

Then we have for $\phi_\alpha \in \mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty K}$ $(\alpha = 1, 2)$,

$$\begin{aligned}
&\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) \\
={}& \mathrm{Vol}(K)\langle \Theta_{\phi_1}(g_1), \Theta_{\phi_2}(g_2)\rangle_{\mathrm{NT}}^K \\
={}& -\mathrm{Vol}(K)\langle \widehat{\Theta}_{\phi_1}(g_1), \widehat{\Theta}_{\phi_2}(g_2)\rangle_{\mathrm{GS}} \\
={}& -\mathrm{Vol}(K)\langle \widehat{Z}_{\phi_1}(g_1) - E(g_1, \phi_1)\widehat{\omega}_K, \widehat{Z}_{\phi_2}(g_2) - E(g_2, \phi_2)\widehat{\omega}_K\rangle_{\mathrm{GS}} \\
={}& -\mathrm{Vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathrm{GS}} + E(g_1, \phi_1)\,\mathrm{Vol}(K)\langle \widehat{\omega}_K, \widehat{\Theta}_{\phi_2}(g_2)\rangle_{\mathrm{GS}} \\
&+ E(g_2, \phi_2)\,\mathrm{Vol}(K)\langle \widehat{\Theta}_{\phi_1}(g_1), \widehat{\omega}_K\rangle_{\mathrm{GS}} + E(g_1, \phi_1)E(g_2, \phi_2)\,\mathrm{Vol}(K)\langle \widehat{\omega}_K, \widehat{\omega}_K\rangle_{\mathrm{GS}},
\end{aligned} \tag{3.12}$$

where the Gillet–Soulé pairings are taken on the model $\mathcal{M}_K$. By Corollary 3.4.2,

$$A(g, \phi) := \mathrm{Vol}(K)\langle \widehat{\omega}_K, \widehat{\Theta}_\phi(g)\rangle_{\mathrm{GS}}$$

is an automorphic form of H', which may depend on K and also the model $\mathcal{M}_K$, since we do not require any canonicality of $p^{-1}_{\mathcal{M}_K}$. Let $C = \mathrm{Vol}(K)\langle \widehat{\omega}_K, \widehat{\omega}_K\rangle_{\mathrm{GS}}$. Then

$$\begin{aligned}
(3.12) ={}& -\mathrm{Vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathrm{GS}} \\
&+ E(g_1, \phi_1)A(g_2, \phi_2) + A(g_1, \phi_1)E(g_2, \phi_2) + CE(g_1, \phi_1)E(g_2, \phi_2).
\end{aligned} \tag{3.13}$$

We assume that ϕ_1 and ϕ_2 are decomposable, and $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for some $v \in \Sigma_{\mathrm{fin}}$. Then $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$ will not intersect on the generic fiber if $g_\alpha \in P'_v H'(\mathbb{A}_F^v)$ $(\alpha = 1, 2)$, and hence

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathrm{GS}} = \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}, \tag{3.14}$$

where the local intersection $\langle -, -\rangle_{v^\circ}$ is taken on the local model $\mathcal{M}_{K;\mathfrak{p}^\circ} := \mathcal{M}_K \times_{\mathcal{O}_E} \mathcal{O}_{E_{\mathfrak{p}^\circ}}$ (resp. $M_{K,\iota^\circ}(\mathbb{C})$) if $v^\circ = \mathfrak{p}^\circ$ is finite (resp. if $v^\circ = \iota^\circ$ is infinite). Combining (3.13) and (3.14), we have for such ϕ_α and g_α $(\alpha = 1, 2)$,

$$\begin{aligned}
\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) ={}& -\mathrm{Vol}(K) \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ} \\
&+ E(g_1, \phi_1)A(g_2, \phi_2) + A(g_1, \phi_1)E(g_2, \phi_2) + CE(g_1, \phi_1)E(g_2, \phi_2).
\end{aligned} \tag{3.15}$$

Chapter 4

Comparison at infinite places

We compare local terms of analytic and arithmetic kernel functions at an archimedean place. Section 4.1 is dedicated to the computation on the analytic side. We calculate certain Whittaker integral and its derivative, following the method of G. Shimura. In 4.2, we introduce a local height function on the hermitian domain in terms of the Kudla–Millson form, and prove an important invariance property of such height. The actual comparison of the derivative of Whittaker integrals and the local height function is accomplished in 4.3.

4.1 Archimedean Whittaker integrals

In this section, we calculate the Whittaker integral $W_T(s,g,\Phi)$ and its derivative (at $s=0$) at an archimedean place. In particular, we fix an archimedean place $\iota : F \to \mathbb{C}$, and a place $\iota' : E \to \mathbb{C}$ over ι, both of which will be suppressed from notations. As in 2.1.2, we identify $H' = H'_\iota$ (resp. $H'' = H''_\iota$) with $\mathrm{U}(n,n)_\mathbb{R}$ (resp. $\mathrm{U}(2n,2n)_\mathbb{R}$). We have the parabolic subgroup $P = P_\iota$ of $H'' = \mathrm{U}(2n,2n)_\mathbb{R}$ as in Notation 2.2.1. Moreover, we have the hermitian space $V = \mathbb{V}_\iota$ of rank $2n$ over $\mathbb{C}$, which is the standard positive definite $2n$-dimensional complex hermitian space, and $\Phi^0 \in \mathcal{S}(V^{2n})$ the Gaussian. We have also a character $\chi = \chi_\iota : \mathbb{C}^\times \to \mathbb{C}^\times$ that is trivial on $\mathbb{R}^\times$. Therefore $\chi(z) = z^{2\ell}/(z\overline{z})^\ell$ for some integer ℓ.

4.1.1 Elementary reduction steps

We study the integral

$$W_T(s,g,\Phi^0) = \int_{\mathrm{Her}_{2n}(\mathbb{C})} \varphi_{\Phi^0,s}(\mathsf{w}n(u)g)\psi_T(n(u))^{-1}\mathrm{d}u \tag{4.1}$$

for $T \in \mathrm{Her}_{2n}(\mathbb{C})$ and $\mathrm{Re}\, s > n$, where $\mathsf{w} = \mathsf{w}_n$ and $\mathrm{d}u$ is the selfdual measure with respect to the additive character $\psi(t) = \exp(2\pi i t)$. We have the formula

$$\omega_\chi([k_1,k_2])\Phi^0 = (\det k_1)^{n+\ell}(\det k_2)^{-n+\ell}\Phi^0$$

for $[k_1,k_2] \in \mathcal{K}$ (*cf.* 2.1.2). Write $g = n(b)m(a)[k_1,k_2]$ under the Iwasawa decomposition. Then

$$\begin{aligned}
(4.1) &= \int_{\mathrm{Her}_{2n}(\mathbb{C})} \left(\omega_\chi\left(\mathsf{w}n(u)n(b)m(a)[k_1,k_2]\right)\Phi^0\right)(0)\lambda_P(\mathsf{w}n(u)n(b)m(a)[k_1,k_2])^s\psi(-\mathrm{tr}\,Tu)\mathrm{d}u \\
&= \psi(\mathrm{tr}\,Tb)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell} \\
&\qquad \int_{\mathrm{Her}_{2n}(\mathbb{C})} \left(\omega_\chi(\mathsf{w}n(u)m(a))\Phi^0\right)(0)\lambda_P(\mathsf{w}n(u)m(a))^s\psi(-\mathrm{tr}\,Tu)\mathrm{d}u.
\end{aligned} \tag{4.2}$$

Since

$$\mathsf{w}n(u)m(a) = \mathsf{w}m(a)n(a^{-1}u\,{}^t\overline{a}^{-1}) = m({}^t\overline{a}^{-1})\mathsf{w}n(a^{-1}u\,{}^t\overline{a}^{-1}),$$

changing variable $\mathrm{d}u = |\det a|_{\mathbb{C}}^{2n} \mathrm{d}(a^{-1} u\, {}^t\overline{a}^{-1})$, we have

$$\begin{aligned}
(4.2) &= \psi(\operatorname{tr} Tb) |\det a|_{\mathbb{C}}^{n-s} \chi(\det a)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell} \\
&\qquad \int_{\mathrm{Her}_{2n}(\mathbb{C})} \left(\omega_\chi(\mathsf{w}n(u))\Phi^0\right)(0) \lambda_P(\mathsf{w}n(u))^s \psi(-\operatorname{tr} {}^t\overline{a}Tau) \mathrm{d}u \\
&= \psi(\operatorname{tr} Tb) |\det a|_{\mathbb{C}}^{n-s} \chi(\det a)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell} W_{{}^t\overline{a}Ta}(s, e, \Phi^0). \qquad (4.3)
\end{aligned}$$

Therefore, we only need to study $W_T(s, e, \Phi^0)$. In what follows, we will not restrict ourselves to the case of even dimension. In other words, V will be the standard positive definite complex hermitian space of dimension $m > 0$. For $T \in \mathrm{Her}_m(\mathbb{C})$, the Whittaker integral $W_T(s, e, \Phi^0)$ is absolutely convergent for $\operatorname{Re} s > \frac{m}{2}$.

Lemma 4.1.1. *For $u \in \mathrm{Her}_m(\mathbb{C})$,*

$$\left(\omega_\chi(\mathsf{w}n(u))\Phi^0\right)(0) = \gamma_V \det(\mathbf{1}_m - iu)^m,$$

where γ_V is the Weil constant.

Proof. By definition,

$$\gamma_V^{-1} \left(\omega_\chi(\mathsf{w}n(u))\Phi^0\right)(0) = \int_{V^m} \left(\omega_\chi(n(u))\Phi^0\right)(x)\mathrm{d}x = \int_{V^m} \psi(\operatorname{tr} uT(x))\Phi^0(x)\mathrm{d}x. \qquad (4.4)$$

Write $u = k \operatorname{diag}[u_1, \dots, u_m]\, {}^t\overline{k}$ with $u_j \in \mathbb{R}$ $(j = 1, \dots, m)$ and $k \in \mathrm{U}(m)_{\mathbb{R}}$. Then

$$\begin{aligned}
(4.4) &= \int_{V^m} \psi\left(\operatorname{tr} k \operatorname{diag}[u_1, \dots, u_m]\, {}^t\overline{k} T(x) k k^{-1}\right) \Phi^0(x)\mathrm{d}x \\
&= \int_{V^m} \psi(\operatorname{tr} \operatorname{diag}[u_1, \dots, u_m] T(xk)) \exp(-2\pi \operatorname{tr} T(x))\mathrm{d}x. \qquad (4.5)
\end{aligned}$$

Changing variable $x \mapsto xk$ and since $\operatorname{tr} T(x) = \operatorname{tr} T(xk)$, we have

$$\begin{aligned}
(4.5) &= \int_{V^m} \exp(2\pi i \operatorname{tr} \operatorname{diag}[u_1, \dots, u_m] T(x) - 2\pi \operatorname{tr} T(x))\mathrm{d}x \\
&= \prod_{j=1}^m \int_V \exp(2\pi i u_j T(x_j) - 2\pi T(x_j))\mathrm{d}x_j. \qquad (4.6)
\end{aligned}$$

Identifying V with $\mathbb{C}^m$ such that $(-,-)$ coincides with the standard hermitian form on $\mathbb{C}^m$, then the selfdual measure $\mathrm{d}x_j$ on V is simply the usual Lebesgue measure $\mathrm{d}\mathbf{x}$ on $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Therefore,

$$\begin{aligned}(4.6) &= \prod_{j=1}^{m} \int_{\mathbb{R}^{2m}} \exp(-\pi(1-iu_j)\|\mathbf{x}\|^2)\mathrm{d}\mathbf{x} \\ &= \prod_{j=1}^{m} \left(\int_{-\infty}^{\infty} \exp(-\pi(1-iu_j)t^2)\mathrm{d}t \right)^{2m} \\ &= \prod_{j=1}^{m} (1-iu_j)^{-m} = \det(\mathbf{1}_m - iu)^{-m}.\end{aligned}$$

Therefore, the lemma follows. □

Lemma 4.1.2. *For $u \in \mathrm{Her}_m(\mathbb{C})$, $\lambda_P(\mathsf{w}n(u)) = \det(\mathbf{1}_m + u^2)^{-1}$.*

Proof. We have the following identities

$$\mathsf{w}n(u) \begin{pmatrix} i\mathbf{1}_m \\ \mathbf{1}_m \end{pmatrix} = \begin{pmatrix} & \mathbf{1}_m \\ -\mathbf{1}_m & \end{pmatrix} \begin{pmatrix} \mathbf{1}_m & u \\ & \mathbf{1}_m \end{pmatrix} \begin{pmatrix} i\mathbf{1}_m \\ \mathbf{1}_m \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m \\ -i\mathbf{1}_m - u \end{pmatrix}.$$

Then,

$$\mathbf{1}_m(-i\mathbf{1}_m - u)^{-1} = -u(\mathbf{1}_m + u^2)^{-1} + i(\mathbf{1}_m + u^2)^{-1}.$$

Therefore, $\lambda_P(\mathsf{w}n(u)) = \det(\mathbf{1}_m + u^2)^{-1}$ that is a positive real number. □

Combining Lemmas 4.1.1 and 4.1.2, we have that for $\mathrm{Re}\, s > \frac{m}{2}$,

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \int_{\mathrm{Her}_m(\mathbb{C})} \psi(-\mathrm{tr}\, Tu) \det(\mathbf{1}_m + iu)^{-s} \det(\mathbf{1}_m - iu)^{-s-m}\mathrm{d}u.$$

We proceed as in [Shi1982, Case II]. We introduce some new notations that may be different from those in [Shi1982]. Set

$$\begin{aligned}\mathrm{Her}_m^+(\mathbb{C}) &= \{x \in \mathrm{Her}_m(\mathbb{C}) \mid x > 0\}; \\ \mathfrak{h}_m &= \{x + iy \mid x \in \mathrm{Her}_m(\mathbb{C}), y \in \mathrm{Her}_m^+(\mathbb{C})\}; \\ \mathfrak{h}'_m &= \{x + iy \mid x \in \mathrm{Her}_m^+(\mathbb{C}), y \in \mathrm{Her}_m(\mathbb{C})\}.\end{aligned}$$

The following lemma is proved in [Shi1982, Section 1].

Lemma 4.1.3 (Siegel)**.** *We have*

1. *For $z \in \mathfrak{h}'_m$ and $\operatorname{Re} s > m-1$, we have*

$$\int_{\mathrm{Her}_m^+(\mathbb{C})} \exp(-\operatorname{tr} zx)(\det x)^{s-m} \mathrm{d}x = \Gamma_m(s)(\det z)^{-s},$$

where $\mathrm{d}x$ is induced from the selfdual measure on $\mathrm{Her}_m(\mathbb{C})$, and

$$\Gamma_m(s) = (2\pi)^{\frac{m(m-1)}{2}} \prod_{j=0}^{m-1} \Gamma(s-j).$$

2. *For $x \in \mathrm{Her}_m(\mathbb{C})$, $b \in \mathrm{Her}_m^+(\mathbb{C})$ and $\operatorname{Re} s > 2m-1$, we have*

$$\Gamma_m(s) \int_{\mathrm{Her}_m(\mathbb{C})} \exp(2\pi i \operatorname{tr} ux) \det(b + 2\pi i u)^{-s} \mathrm{d}u = \begin{cases} \exp(-\operatorname{tr} xb)(\det x)^{s-m} & x \in \mathrm{Her}_m^+(\mathbb{C}); \\ 0 & \textit{if not.} \end{cases}$$

By Lemma 4.1.3 (1), for $\operatorname{Re} s > m-1$,

$$\begin{aligned}
&\gamma_V^{-1} W_T(s, e, \Phi^0) \\
&= \int_{\mathrm{Her}_m(\mathbb{C})} \psi(-\operatorname{tr} Tu) \frac{1}{\Gamma_m(s)} \int_{\mathrm{Her}_m^+(\mathbb{C})} \exp(-\operatorname{tr}(\mathbf{1}_m + iu)x)(\det x)^{s-m} \det(\mathbf{1}_m - iu)^{-s-m} \mathrm{d}x\mathrm{d}u \\
&= \frac{1}{\Gamma_m(s)} \int_{\mathrm{Her}_m^+(\mathbb{C})} \exp(-\operatorname{tr} x)(\det x)^{s-m} \int_{\mathrm{Her}_m(\mathbb{C})} \exp(-i \operatorname{tr}(x + 2\pi T)u) \det(\mathbf{1}_m - iu)^{-s-m} \mathrm{d}u\mathrm{d}x.
\end{aligned} \tag{4.7}$$

Applying Lemma 4.1.3 (2) to $(\mathbf{1}_m, x + 2\pi, s + m)$, and changing variable $u \mapsto -\frac{u}{2\pi}$, we have

$$(4.7) = \frac{1}{\Gamma_m(s)} \int_{x>0, x+2\pi T>0} \exp(-\operatorname{tr} x)(\det x)^{s-m} \frac{(2\pi)^{m^2}}{\Gamma_m(s+m)} \exp(-\operatorname{tr}(x + 2\pi T)) \det(x + 2\pi T)^s \mathrm{d}x. \tag{4.8}$$

In [Shi1982, (1.26)], the author introduced the function

$$\eta(g, h; \alpha, \beta) = \int_{x>-h,\ x>h} \exp(-\operatorname{tr} gx) \det(x+h)^{\alpha-m} \det(x-h)^{\beta-m} \mathrm{d}x$$

for $g \in \mathrm{Her}_m^+(\mathbb{C}), h \in \mathrm{Her}_m(\mathbb{C})$, and $\mathrm{Re}\,\alpha \gg 0$, $\mathrm{Re}\,\beta \gg 0$. Changing variable $x \mapsto \frac{x}{\pi} + T$, we have

$$\begin{aligned}(4.8) &= \frac{(2\pi)^{m^2}\pi^{2ms}}{\Gamma_m(s)\Gamma_m(s+m)} \int_{x>-T, x>T} \exp(-\operatorname{tr} 2\pi x) \det(x+T)^s \det(x-T)^{s-m} \mathrm{d}x \\ &= \frac{(2\pi)^{m^2}\pi^{2ms}}{\Gamma_m(s)\Gamma_m(s+m)} \eta(2\pi \mathbf{1}_m, T; s+m, s). \end{aligned} \tag{4.9}$$

In what follows, we assume that T is nonsingular with $\operatorname{sign} T = (p,q)$ for $p+q=m$. Write

$$T = k \operatorname{diag}[t_1, \dots, t_p, -t_{p+1}, \dots, -t_m]\, {}^t\overline{k}$$

with $k \in \mathrm{U}(m)_{\mathbb{R}}$ and $t_j \in \mathbb{R}_{>0}$. Let $a = k \operatorname{diag}[\sqrt{t_1}, \dots, \sqrt{t_m}]$. Then $T = a\varepsilon_{p,q}\, {}^t\overline{a}$, where

$$\varepsilon_{p,q} = \begin{pmatrix} \mathbf{1}_p & \\ & -\mathbf{1}_q \end{pmatrix}.$$

It is easy to see that

$$\eta(g, T; \alpha, \beta) = |\det T|^{\alpha+\beta-m} \eta(a^* g a, \varepsilon_{p,q}; \alpha, \beta); \tag{4.10}$$

$$\eta(g, \varepsilon_{p,q}; \alpha, \beta) = 2^{m(\alpha+\beta-m)} \exp(-\operatorname{tr} g) \zeta_{p,q}(2g; \alpha, \beta). \tag{4.11}$$

We recall the definition of $\zeta_{p,q}(g; \alpha, \beta)$ introduced in [Shi1982, (4.16)]. Let

$$\varepsilon_p = \begin{pmatrix} \mathbf{1}_p & \\ & \mathbf{0}_q \end{pmatrix}; \qquad \varepsilon'_q = \begin{pmatrix} \mathbf{0}_p & \\ & \mathbf{1}_q \end{pmatrix}.$$

Then for $g \in \mathrm{Her}_m^+(\mathbb{C})$, and $\mathrm{Re}\,\alpha \gg 0$, $\mathrm{Re}\,\beta \gg 0$,

$$\zeta_{p,q}(g; \alpha, \beta) = \int_{X_{p,q}} \exp(-\operatorname{tr} gx) \det(x + \varepsilon_p)^{\alpha-m} \det(x + \varepsilon'_q)^{\beta-m} \mathrm{d}x,$$

where

$$X_{p,q} := \{x \in \mathrm{Her}_m(\mathbb{C}) \mid x + \varepsilon_p > 0, x + \varepsilon'_q > 0\}$$

with the measure induced from the selfdual one on $\mathrm{Her}_m(\mathbb{C})$. In particular, $X_{m,0} = \mathrm{Her}_m^+(\mathbb{C})$.

4.1.2 Analytic continuation

Following [Shi1982, (4.17)], we set

$$\omega_{p,q}(g;\alpha,\beta)=\Gamma_q(\alpha-p)^{-1}\Gamma_p(\beta-q)^{-1}(\det{}^+\varepsilon_{p,q}g)^{\beta-q/2}(\det{}^-\varepsilon_{p,q}g)^{\alpha-p/2}\zeta_{p,q}(g;\alpha,\beta), \tag{4.12}$$

where for a nonsingular element $h\in\mathrm{Her}_m(\mathbb{C})$, $\det^+ h$ (resp. $\det^- h$) is the absolute value of the product of all positive (resp. negative) (real) eigenvalues of h if they exist; 1 otherwise. It is proved in [Shi1982, Section 4] that $\omega_{p,q}(g;\alpha,\beta)$ has a holomorphic continuation in (α,β) to the entire $\mathbb{C}^2$, and satisfies the following functional equation

$$\omega_{p,q}(g;m-\beta,m-\alpha)=\omega_{p,q}(g;\alpha,\beta).$$

Lemma 4.1.4. *If $p=m$ and $q=0$, then $\omega_{m,0}(g;m,\beta)=\omega_{m,0}(g;\alpha,0)=1$.*

Proof. The integral

$$\zeta_{m,0}(g;m,\beta)=\int_{\mathrm{Her}_m^+(\mathbb{C})}\exp(-\operatorname{tr}gx)(\det x)^{\beta-m}\mathrm{d}x$$

is absolutely convergent for $\operatorname{Re}\beta>m-1$, and equal to $\Gamma_m(\beta)(\det g)^{-\beta}$ by Lemma 4.1.3 (1). Therefore, $\omega_{m,0}(g;m,\beta)=1$, which confirms the lemma by the functional equation. □

Proposition 4.1.5. *Suppose $T\in\mathrm{Her}_m(\mathbb{C})$ is nonsingular with $\operatorname{sign}T=(p,q)$. Then*

1. $\mathrm{ord}_{s=0}W_T(s,e,\Phi^0)\geq q$; *and*

2. *If T is positive definite, i.e. $p=m$ and $q=0$, then*

$$W_T(0,e,\Phi^0)=\gamma_V\frac{(2\pi)^{m^2}}{\Gamma_m(m)}\exp(-2\pi\operatorname{tr}T).$$

Proof. 1. Combining (4.9), (4.10), (4.11) and (4.12), we have

$$\begin{aligned}\gamma_V^{-1}W_T(s,e,\Phi^0)=&\frac{\Gamma_q(m+s-p)\Gamma_p(s-q)}{\Gamma_m(s)\Gamma_m(s+m)}(2\pi)^{m^2+2ms}|\det T|^{2s}\exp(-2\pi\operatorname{tr}{}^t\overline{a}a)\\&(\det{}^+4\pi T)^{q/2-s}(\det{}^-4\pi T)^{p/2-m-s}\omega_{p,q}(4\pi\,{}^t\overline{a}a;m+s,s).\end{aligned}\tag{4.13}$$

All terms except the Gamma factors, are holomorphic for all $s\in\mathbb{C}$. Since

$$\frac{\Gamma_q(m+s-p)\Gamma_p(s-q)}{\Gamma_m(s)\Gamma_m(s+m)}=\frac{(2\pi)^{-pq-\frac{m(m-1)}{2}}}{\Gamma(s)\cdots\Gamma(s-q+1)\times\Gamma(s+m)\cdots\Gamma(s+m-p+1)},$$

we have

$$\mathrm{ord}_{s=0} W_T(s, e, \Phi^0) \geq -\mathrm{ord}_{s=0}\Gamma(s)\cdots\Gamma(s-q+1) = q.$$

2. If T is positive definite, then $\mathrm{tr}\,{}^t\overline{a}a = \mathrm{tr}\,T$. By (4.13) and Lemma 4.1.4, we have

$$\gamma_V^{-1} W_T(0, e, \Phi^0) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)} \exp(-2\pi\,\mathrm{tr}\,T)\omega_{m,0}(4\pi\,{}^t\overline{a}a; m, 0) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)} \exp(-2\pi\,\mathrm{tr}\,T).$$

$\square$

4.1.3 First-order derivatives

By Proposition 4.1.5 (1), the T-th coefficient will not contribute to the analytic kernel function $E'(0, g, \Phi)$ if $\mathrm{sign}\,T = (p, q)$ with $q \geq 2$. Therefore, we focus on the case where $q = 1$, and study the functions $\zeta_{m-1,1}(g; \alpha, \beta)$ and $\omega_{m-1,1}(g; \alpha, \beta)$[1].

We assume that

$$g = \begin{pmatrix} \mathbf{a} & \\ & b \end{pmatrix}, \qquad \mathbf{a} \in \mathrm{Her}_{m-1}^{+}(\mathbb{C}), b \in \mathbb{R}_{>0}.$$

We write elements in $X_{m-1,1}$ in the following form

$$\begin{pmatrix} x & z \\ {}^t\overline{z} & y \end{pmatrix}, \qquad x \in \mathrm{Her}_{m-1}(\mathbb{C}), y \in \mathbb{R}, z \in \mathrm{Mat}_{m-1,1}(\mathbb{C}).$$

Then by [Shi1982, Page 288],

$$\begin{aligned} X_{m-1,1} &= \{(x, y, z) \mid x > 0, y > 0, x + \mathbf{1}_{m-1} > z y^{-1}\,{}^t\overline{z}, y + 1 > {}^t\overline{z} x^{-1} z\} \\ &= \{(x, y, z) \mid x + \mathbf{1}_{m-1} > 0, y + 1 > 0, x > z(y+1)^{-1}\,{}^t\overline{z}, y > {}^t\overline{z}(x + \mathbf{1}_{m-1})^{-1} z\}. \end{aligned}$$

We have

$$\zeta_{m-1,1}(g; \alpha, \beta) = \int_{X_{m-1,1}} \exp(-\,\mathrm{tr}\,\mathbf{a}x - by) \det \begin{pmatrix} x + \mathbf{1}_{m-1} & z \\ {}^t\overline{z} & z \end{pmatrix}^{\alpha - m} \det \begin{pmatrix} x & z \\ {}^t\overline{z} & y + 1 \end{pmatrix}^{\beta - m} \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{4.14}$$

where we apply the selfdual measure $\mathrm{d}x$ on $\mathrm{Her}_{m-1}(\mathbb{C})$, the Lebesgue measure $\mathrm{d}y$ on $\mathbb{R}$, and the

[1] Recall that from our notations, the meaning of the letter g in $\zeta_{m-1,1}(g; \alpha, \beta)$ and $\omega_{m-1,1}(g; \alpha, \beta)$ is different from that in $E'(0, g, \Phi)$.

measure $\mathrm{d}z$ that is 2^{m-1} times the Lebesgue measure on $\mathrm{Mat}_{m-1,1}(\mathbb{C})$. We make change of variables as in [Shi1982, Page 289] as follows. Put

$$f = (x + \mathbf{1}_{m-1})^{-1/2} z (y+1)^{-1/2}.$$

Then $\mathbf{1}_{m-1} - {}^t\overline{f} > 0$. Put

$$r = (1 - {}^t\overline{f} f)^{1/2}; \quad s = (\mathbf{1}_{m-1} - f\,{}^t\overline{f})^{1/2}; \quad w = s^{-1} f = f r^{-1}; \quad u = x - w\,{}^t\overline{w}; \quad v = y - {}^t\overline{w} w.$$

Then the map $(x, y, z) \mapsto (u, v, w)$ maps $X_{m-1,1}$ bijectively onto $Y = \mathrm{Her}^+_{m-1}(\mathbb{C}) \times \mathbb{R}_{>0} \times \mathrm{Mat}_{m-1,1}(\mathbb{C})$, and the Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det(\mathbf{1}_{m-1} + x)(1+y)^{m-1}(1 + {}^t\overline{w} w)^{-m}$$

for the measure $\partial(u, v, w)$ on Y induced from that on $\mathrm{Her}_{m-1}(\mathbb{C}) \times \mathbb{R} \times \mathrm{Mat}_{m-1,1}(\mathbb{C})$ as an open subset. Since

$$\det \begin{pmatrix} x + \mathbf{1}_{m-1} & z \\ {}^t\overline{z} & y \end{pmatrix} = \det(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w}) v \det(\mathbf{1}_{m-1} w\,{}^t\overline{w})^{-1};$$

$$\det \begin{pmatrix} x & z \\ {}^t\overline{z} & y+1 \end{pmatrix} = (v + 1 + {}^t\overline{w} w)(\det u) \det(\mathbf{1}_{m-1} w\,{}^t\overline{w})^{-1},$$

we obtain that

$$\begin{aligned}
(4.14) = &\int_Y \exp\left(-\operatorname{tr}(\mathbf{a} u + \mathbf{a} w\,{}^t\overline{w}) - (bv + b\,{}^t\overline{w} w)\right) \det(\mathbf{1}_{m-1} + w\,{}^t\overline{w})^{m-\alpha-\beta} \\
&\quad \det(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})^{\alpha-m+1} (\det u)^{\beta-m} (v + 1 + {}^t\overline{w} w)^{\beta-1} v^{\alpha-m} \mathrm{d}u\mathrm{d}v\mathrm{d}w \\
= &\int_{\mathrm{Mat}_{m-1,1}(\mathbb{C})} \exp(-\operatorname{tr} \mathbf{a} w\,{}^t\overline{w} - b\,{}^t\overline{w} w) \zeta_{1,0}(b(1 + {}^t\overline{w} w); \beta, \alpha - m + 1) \\
&\quad \int_{\mathrm{Her}^+_{m-1}(\mathbb{C})} \exp(-\operatorname{tr} \mathbf{a} u) \det(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})^{\alpha-m-1} (\det u)^{\beta-m} \mathrm{d}u\mathrm{d}w.
\end{aligned} \tag{4.15}$$

By (4.10), (4.11) and (4.12), we have, on the other hand, that

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \frac{(2\pi)^{m^2+2ms} |\det T|^{2s}}{\Gamma_m(s)\Gamma_m(s+m)} \exp(-2\pi \operatorname{tr} {}^t\overline{a} a) \zeta_{m-1,1}(4\pi\,{}^t\overline{a} a; m+s, s). \tag{4.16}$$

Assume that $T = k \operatorname{diag}[a_1, \ldots, a_{m-1}, -b]\,{}^t\overline{k}$ with $a_1, \ldots, a_{m-1}, b \in \mathbb{R}_{>0}$ and $k \in \mathrm{U}(m)_{\mathbb{R}}$. Then

${}^t\overline{a}a = \mathrm{diag}[a_1, \ldots, a_{m-1}, b]$. By (4.16) and (4.13),

$$\begin{aligned}&\gamma_V^{-1} W_T'(0, e, \Phi^0)\\ &= \lim_{s\to 0} \frac{(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} \exp\left(-2\pi(a_1 + \cdots + a_{m-1} + b)\right) \zeta_{m-1,1}(4\pi\,\mathrm{diag}[a_1, \ldots, a_{m-1}, b]; m, s).\end{aligned} \tag{4.17}$$

Plugging (4.15) with $(\alpha, \beta) = (m, s)$,

$$\begin{aligned}(4.17) = &\lim_{s\to 0} \frac{2^{m-1}(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} \exp\left(-2\pi(a_1 + \cdots + a_{m-1} + b)\right)\\ &\int_{\mathbb{C}^{m-1}} \exp\left(-4\pi[(a_1 + a_m)w_1\overline{w_1} + \cdots + (a_{m-1} + a_m)w_{m-1}\overline{w_{m-1}}]\right) \zeta_{1,0}(4\pi b(1 + {}^t\overline{w}w); 0, 1)\\ &\int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} \exp\left(-4\pi\,\mathrm{tr}\,\mathrm{diag}[a_1, \ldots, a_{m-1}]u\right) \det(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})(\det u)^{s-m} \mathrm{d}u\mathrm{d}w_1 \cdots \mathrm{d}w_{m-1}.\end{aligned} \tag{4.18}$$

It is easy to see that

$$\zeta_{1,0}(4\pi b(1 + {}^t\overline{w}w); 0, 1) = -\exp\left(4\pi b(1 + {}^t\overline{w}w)\right) \mathrm{Ei}\left(-4\pi b(1 + {}^t\overline{w}w)\right), \tag{4.19}$$

where Ei is the exponential integral

$$\mathrm{Ei}(z) = -\int_1^\infty \frac{\exp(zt)}{t} \mathrm{d}t.$$

We evaluate the inside integral, *i.e.* the one over $\mathrm{Her}_{m-1}^+(\mathbb{C})$. Temporarily let $g_0 = 4\pi\,\mathrm{diag}[a_1, \ldots, a_{m-1}]$, and consider the integral

$$\int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} \exp(-\,\mathrm{tr}\,ug_0) \det(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})(\det u)^{s-m} \mathrm{d}u. \tag{4.20}$$

Define a differential operator

$$\Delta = \det\left(\frac{\partial}{\partial g_{jk}}\right)_{j,k=1}^{m-1}.$$

Then

$$\Delta \exp(-\,\mathrm{tr}\,ug) = (-1)^{m-1}(\det u)\exp(-\,\mathrm{tr}\,ug).$$

Therefore,

$$
\begin{aligned}
(4.20) &= \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \\
&\qquad \int_{\mathrm{Her}^+_{m-1}(\mathbb{C})} \exp\left(-\operatorname{tr}(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \det(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})(\det x)^{s-m} \mathrm{d}u \\
&= (-1)^{m-1} \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \\
&\qquad \int_{\mathrm{Her}^+_{m-1}(\mathbb{C})} \Delta\,|_{g=g_0} \exp\left(-\operatorname{tr}(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) (\det x)^{s-m} \mathrm{d}u.
\end{aligned}
\tag{4.21}
$$

We exchange Δ and the integration by analytic continuation. Then

$$
\begin{aligned}
(4.21) &= (-1)^{m-1} \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \\
&\qquad \Delta\,|_{g=g_0} \int_{\mathrm{Her}^+_{m-1}(\mathbb{C})} \exp\left(-\operatorname{tr}(u + \mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) (\det x)^{s-m} \mathrm{d}u \\
&= (-1)^{m-1} \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \Delta\,|_{g=g_0} \left(\exp\left(-\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) \zeta_{m-1}(g; m-1, s-1)\right) \\
&= (-1)^{m-1} \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \Delta\,|_{g=g_0} \left(\exp\left(-\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) (\det g)^{1-s} \Gamma_{m-1}(s-1)\right) \\
&= (-1)^{m-1} \Gamma_{m-1}(s-1) \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \Delta\,|_{g=g_0} \left(\exp\left(-\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) (\det g)^{1-s}\right).
\end{aligned}
\tag{4.22}
$$

Plugging (4.19) and (4.22) into (4.18), we obtain

$$
\begin{aligned}
(4.18) &= \lim_{s\to 0} \frac{\Gamma_{m-1}(s-1)(-2)^{m-1}(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} \exp(-2\pi \operatorname{tr} T) \\
&\qquad \int_{\mathbb{C}^{m-1}} \exp\left(-4\pi(a_1 w_1 \overline{w_1} + \cdots + a_{m-1} w_{m-1} \overline{w_{m-1}})\right) \\
&\qquad \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \Delta\,|_{g=g_0} \left(\exp\left(-\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) (\det g)^{1-s}\right) \\
&\qquad (-\operatorname{Ei})\left(-4\pi b(1 + {}^t\overline{w}w)\right) \mathrm{d}w_1 \cdots \mathrm{d}w_{m-1} \\
&= \frac{(2\pi)^{m^2}(-2)^{m-1}}{\Gamma_m(m)(2\pi)^{m-1}} \exp(-2\pi \operatorname{tr} T) \int_{\mathbb{C}^{m-1}} \exp\left(-4\pi(a_1 w_1 \overline{w_1} + \cdots + a_{m-1} w_{m-1} \overline{w_{m-1}})\right) \\
&\qquad \exp\left(\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g_0\right) \Delta\,|_{g=g_0} \left(\exp\left(-\operatorname{tr}(\mathbf{1}_{m-1} + w\,{}^t\overline{w})g\right) (\det g)\right) \\
&\qquad (-\operatorname{Ei})\left(-4\pi b(1 + {}^t\overline{w}w)\right) \mathrm{d}w_1 \cdots \mathrm{d}w_{m-1}.
\end{aligned}
\tag{4.23}
$$

To compare with the local height later, we make a change of variables. Let

$$
D_{m-1} = \{z = (z_1, \ldots, z_{m-1}) \in \mathbb{C}^{m-1} \mid z\overline{z} := z_1\overline{z_1} + \cdots + z_{m-1}\overline{z_{m-1}} < 1\}
$$

be the open unit disc in $\mathbb{C}^{m-1}$. Then the map

$$w_j = \frac{z_j}{(1 - z\overline{z})^{1/2}}, \qquad j = 1, \ldots, m-1 \tag{4.24}$$

is a homeomorphism from $\mathbb{C}^{m-1}$ to D_{m-1} as real manifolds. To calculate the Jacobian, let $w_j = u_j + v_j i$ and $z_j = x_j + y_j i$ be the corresponding real and imaginary parts. Then

$$\begin{aligned}
\frac{\partial u_j}{\partial x_k} &= \frac{x_j x_k}{(1 - z\overline{z})^{3/2}}, \qquad k \neq j; \\
\frac{\partial u_j}{\partial x_j} &= \frac{x_j^2}{(1 - z\overline{z})^{3/2}} + \frac{1}{(1 - z\overline{z})^{1/2}}; \\
\frac{\partial u_j}{\partial y_k} &= \frac{x_j y_k}{(1 - z\overline{z})^{3/2}}; \\
\frac{\partial v_j}{\partial y_k} &= \frac{y_j y_k}{(1 - z\overline{z})^{3/2}}, \qquad k \neq j; \\
\frac{\partial v_j}{\partial y_j} &= \frac{y_j^2}{(1 - z\overline{z})^{3/2}} + \frac{1}{(1 - z\overline{z})^{1/2}}; \\
\frac{\partial v_j}{\partial x_k} &= \frac{y_j x_k}{(1 - z\overline{z})^{3/2}}.
\end{aligned}$$

In Lemma 4.1.6 below, we let $n = 2m+2$, $\epsilon = 1 - z\overline{z}$ and $c = {}^t(c_1, \ldots, c_{2m-2})$ with $c_j = x_j$, $c_{m+1-j} = y_j$ for $j = 1, \ldots, m-1$. Then

$$\begin{aligned}
&\frac{\partial(u_1, v_1, \ldots, u_{m-1}, v_{m-1})}{\partial(x_1, y_1, \ldots, x_{m-1}, y_{m-1})} \\
&= \frac{\partial(u_1, \ldots, u_{m-1}; v_1, \ldots, v_{m-1})}{\partial(x_1, \ldots, x_{m-1}; y_1, \ldots, y_{m-1})} \\
&= (1 - z\overline{z})^{-3(m-1)} \det\left((1 - z\overline{z})\,\mathbf{1}_{2m-2} + c\,{}^t\overline{c}\right) \\
&= (1 - z\overline{z})^{-3(m-1)} (1 - z\overline{z})^{2m-3} \left(1 - z\overline{z} + x_1^2 + \cdots + x_{m-1}^2 + y_1^2 + \cdots + y_{m-1}^2\right) \\
&= (1 - z\overline{z})^{-m}.
\end{aligned} \tag{4.25}$$

Lemma 4.1.6. *Let* $c \in \mathrm{Mat}_{n\times 1}(\mathbb{C})$. *Then*

1. $\det\left(\mathbf{1}_n + c\,{}^t\overline{c}\right) = 1 + {}^t\overline{c}c$;
2. *For* $\epsilon > 0$, $\det\left(\epsilon\mathbf{1}_n + c\,{}^t\overline{c}\right) = \epsilon^{n-1}\left(\epsilon + {}^t\overline{c}c\right)$.

Proof. 1. It is [Shi1982, Lemma 2.2]. Since it is not difficult, we will give a proof here for completeness, following Shimura. We claim that $\det\left(\mathbf{1}_n + sc\,{}^t\overline{c}\right) = 1 + s\,{}^t\overline{c}c$ for all $c \in \mathbb{R}$. Since they are both polynomials in s, we need only to prove for $s<0$. We have

$$\begin{pmatrix} \mathbf{1}_n & -\sqrt{-s}c \\ & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1}_n & \sqrt{-s}c \\ \sqrt{-s}\,{}^t\overline{c} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1}_n & \\ -\sqrt{-s}\,{}^t\overline{c} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n + sc\,{}^t\overline{c} & \\ & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} \mathbf{1}_n & \\ -\sqrt{-s}\,{}^t\overline{c} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1}_n & \sqrt{-s}c \\ \sqrt{-s}\,{}^t\overline{c} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1}_n & -\sqrt{-s}c \\ & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & \\ & 1+s\,{}^t\overline{c}c \end{pmatrix}.$$

Therefore, $\det\left(\mathbf{1}_n + sc\,{}^t\overline{c}\right) = 1 + s\,{}^t\overline{c}c$.

2. It follows from (1) immediately.

□

Now we write the Lebesgue measure $\mathrm{d}z_1\cdots\mathrm{d}z_{m-1}$ in the differential form of degree $(m-1,m-1)$ on D_{m-1} which is

$$\mathrm{d}z_1\cdots\mathrm{d}z_{m-1} = \frac{1}{(-2i)^{m-1}}\Omega,$$

where

$$\Omega = \bigwedge_{j=1}^{m-1}\left(\mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}\right). \tag{4.26}$$

Here, we view $\mathrm{d}z_j$ as a $(1,0)$-form, not the Lebesgue measure. By (4.25), we have

$$\begin{aligned}(4.23) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)(2\pi i)^{m-1}} \exp\left(-2\pi\,\mathrm{tr}\,T\right)\int_{D_{m-1}} &\exp\left(-4\pi(a_1w_1\overline{w_1}+\cdots+a_{m-1}w_{m-1}\overline{w_{m-1}})\right)(1-z\overline{z})^{-m} \\ &\exp\left(\mathrm{tr}\left(\mathbf{1}_{m-1}+w\,{}^t\overline{w}\right)g_0\right)\Delta\mid_{g=g_0}\left(\exp\left(-\,\mathrm{tr}\left(\mathbf{1}_{m-1}+w\,{}^t\overline{w}\right)g\right)(\det g)\right) \\ &(-\,\mathrm{Ei})\left(-4\pi b\left(1+{}^t\overline{w}w\right)\right)\Omega,\end{aligned} \tag{4.27}$$

where w_j are as in (4.24). The final step is accomplished by the following lemma.

Lemma 4.1.7. *For $g_0 = 4\pi\,\mathrm{diag}[a_1,\ldots,a_{m-1}]$,*

$$\begin{aligned}&\Delta\mid_{g=g_0}\left(\exp\left(-\,\mathrm{tr}\left(\mathbf{1}_{m-1}+w\,{}^t\overline{w}\right)g\right)(\det g)\right) \\ &= \exp\left(-\,\mathrm{tr}\left(\mathbf{1}_{m-1}+w\,{}^t\overline{w}\right)g\right) \\ &\quad \sum_{1\le s_1<\cdots<s_t\le m-1}(-4\pi)^t(m-1-t)!(a_{s_1}\cdots a_{s_t})\left(1+w_{s_1}\overline{w_{s_1}}+\cdots+w_{s_t}\overline{w_{s_t}}\right),\end{aligned}$$

where the sum is taken over all subsets of $\{1, \ldots, m-1\}$.

Proof. Let

$$u_{jk} = -\left(1 + w_j \overline{w_k}\right); \qquad g = (g_{jk})_{j,k=1}^{m-1}$$

be the variables in matrices. For short, we use $|g|$ to indicate the determinant of a square matrix g. For subsets $I, J \subset \{1, \ldots, m-1\}$ of the same cardinality, we denote by $g_{J,K}$ (resp. $g^{J,K}$) the (square) matrix obtained by keeping (resp. discarding) the rows indexed in J and the columns indexed in K. Therefore, $g^{J,K} = g_{\overline{J},\overline{K}}$, where $\overline{J}$ (resp. $\overline{K}$) is the complement set $\{1, \ldots, m-1\} - J$ (resp. $\{1, \ldots, m-1\} - K$). Let $\mathfrak{S}_{m-1}$ be the group of $(m-1)$-permutations. For $\sigma \in \mathfrak{S}_{m-1}$ and a subset $J = \{j_1 < \cdots < j_t\} \subset \{1, \ldots, m-1\}$, let $\epsilon_J(\sigma) \in \{\pm 1\}$ be a factor that depends only on J and σ. This factor comes from the combinatorics in taking successive partial derivatives. In later calculation, we only need to know its value in the case where σ maps J to itself. Then, if we let σ_J be the restriction of σ to J, we have $\epsilon_J(\sigma) = (-1)^{|\sigma_J|}$.

We compute that

$$\begin{aligned}
\frac{\partial}{\partial g_{1,\sigma(1)}} \left(\exp\left(\operatorname{tr} ug\right)|g|\right) &= u_{\sigma(1),1} \exp\left(\operatorname{tr} ug\right)|g| + \epsilon_{\{1\}}(\sigma) \exp(\operatorname{tr} ug) \left|g^{\{1\},\{\sigma(1)\}}\right|; \\
\frac{\partial}{\partial g_{2,\sigma(2)}} \frac{\partial}{\partial g_{1,\sigma(1)}} \left(\exp(\operatorname{tr} ug)|g|\right) &= u_{\sigma(2),2} u_{\sigma(1),1} \exp(\operatorname{tr} ug)|g| + \epsilon_{\{2\}}(\sigma) u_{\sigma(1),1} \exp(\operatorname{tr} ug) \left|g^{\{2\},\{\sigma(2)\}}\right| \\
&\quad + \epsilon_{\{1\}}(\sigma) u_{\sigma(2),2} \exp(\operatorname{tr} ug) \left|g^{\{1\},\{\sigma(1)\}}\right| \\
&\quad + \epsilon_{\{1,2\}}(\sigma) \exp(\operatorname{tr} ug) \left|g^{\{1,2\},\{\sigma(1),\sigma(2)\}}\right|.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&\frac{\partial}{\partial g_{m-1,\sigma(m-1)}} \cdots \frac{\partial}{\partial g_{1,\sigma(1)}} \left(\exp(\operatorname{tr} ug)|g|\right) \\
&= \sum_{1 \le j_1 < \cdots < j_t \le m-1} \epsilon_{\{j_1, \ldots, j_t\}}(\sigma) u_{\sigma(s_{m-1-t}), s_{m-1-t}} \cdots u_{\sigma(s_1), s_1} \exp(\operatorname{tr} ug) \left|g^{\{j_1, \ldots, j_t\},\{\sigma(j_1), \ldots, \sigma(j_t)\}}\right|,
\end{aligned}$$

where $\{s_1 < \cdots < s_{m-1-t}\}$ is the complement of $\{j_1, \ldots, j_t\}$. Summing over σ, we have

$$\begin{aligned}
\Delta \left.\right|_{g=g_0} \left(\exp(\operatorname{tr} ug)|g|\right) = \exp(\operatorname{tr} ug_0) &\sum_{\sigma \in \mathfrak{S}_{m-1}} (-1)^{|\sigma|} \sum_{1 \le j_1 < \cdots < j_t \le m-1} \\
&\epsilon_{\{j_1, \ldots, j_t\}}(\sigma) u_{\sigma(s_{m-1-t}), s_{m-1-t}} \cdots u_{\sigma(s_1), s_1} \left|g_0^{\{j_1, \ldots, j_t\},\{\sigma(j_1), \ldots, \sigma(j_t)\}}\right|.
\end{aligned}$$

Changing the order of summation, since g_0 is diagonal, we have

$$\begin{aligned}
&\Delta\,|_{g=g_0}\,(\exp(\operatorname{tr} ug)|g|)\\
&=\exp(\operatorname{tr} ug_0)\sum_{J=\{j_1<\cdots<j_t\}}\sum_{\sigma(J)=J}(-1)^{|\sigma|}(-1)^{|\sigma_J|}u_{\sigma(s_{m-1-t}),s_{m-1-t}}\cdots u_{\sigma(s_1),s_1}\left|g_0^{J,J}\right|\\
&=\exp(\operatorname{tr} ug_0)\sum_{J=\{j_1<\cdots<j_t\}}t!\left|g_0^{J,J}\right|\sum_{\sigma':\overline{J}\to\overline{J}}(-1)^{|\sigma'|}u_{\sigma(s_{m-1-t}),s_{m-1-t}}\cdots u_{\sigma(s_1),s_1}\\
&=\exp(\operatorname{tr} ug_0)\sum_{J=\{j_1<\cdots<j_t\}}t!\left|g_0^{J,J}\right|\left|u^{J,J}\right|\\
&=\exp(\operatorname{tr} ug_0)\sum_{J'=\{s_1<\cdots<s_t\}}(m-1-t)!\,|(g_0)_{J',J'}|\,|u_{J',J'}|\,.
\end{aligned}$$

The lemma follows by Lemma 4.1.6 (1). □

In conclusion, combining (4.27), we obtain the following proposition.

Proposition 4.1.8. *For $T=k\operatorname{diag}[a_1,\dots,a_{m-1},-b]\,{}^t\overline{k}$ of signature $(m-1,1)$ as above, we have*

$$\begin{aligned}
&W_T'(0,e,\Phi^0)\\
&=\gamma_V\frac{(2\pi)^{m^2}}{\Gamma_m(m)(2\pi i)^{m-1}}\exp(-2\pi\operatorname{tr}T)\int_{D_{m-1}}\exp\left(-4\pi(a_1w_1\overline{w_1}+\cdots+a_{m-1}w_{m-1}\overline{w_{m-1}})\right)\\
&\quad\sum_{1\le s_1<\cdots<s_t\le m-1}(-4\pi)^t(m-1-t)!(a_{s_1}\cdots a_{s_t})\left(1+w_{s_1}\overline{w_{s_1}}+\cdots+w_{s_t}\overline{w_{s_t}}\right)\\
&\quad(-\operatorname{Ei})\left(-4\pi b(1+w^*w)\right)(1-z\overline{z})^{-m}\,\Omega,
\end{aligned}$$

where w_j are functions in z as in (4.24), and Ω (4.26) is the volume form in z.

4.2 Archimedean local height

In this section, we introduce a notion of height on the symmetric domain that will eventually contribute to the local height pairing at an archimedean place. We also prove some properties of such height. A basic reference for archimedean Green currents and height pairing is [Sou1992, Chapter II]. We keep the notations in 4.1. We fix an integer $m\ge 2$.

4.2.1 Green currents

Let $V' \simeq \mathbb{C}^m$ be the complex hermitian space with the form

$$(z', z) = z'_1\overline{z_1} + \cdots + z'_{m-1}\overline{z_{m-1}} - z'_m\overline{z_m}; \qquad z = (z_1, \ldots, z_m), z' = (z'_1, \ldots, z'_m) \in \mathbb{C}^m.$$

In particular, the signature of V' is $(m-1, 1)$. The symmetric hermitian domain $\mathcal{D}$ of $\mathrm{U}(V')$, introduced in 3.1.1, can be identified with the $(m-1)$-dimensional open complex unit disc D_{m-1} via the map

$$z = [z_1 : \cdots : z_m] \in \mathcal{D} \mapsto \left(\frac{z_1}{z_m}, \ldots, \frac{z_{m-1}}{z_m}\right) \in D_{m-1}.$$

In what follows, we will not distinguished between $\mathcal{D}$ and D_{m-1}.

Given any $x \in V'^r$ $(1 \leq r \leq m-1)$ with nonsingular moment matrix $T(x)$, let D_x be the subspace of D_{m-1} consisting of lines perpendicular to all components in x. Then D_x is nonempty if and only if $T(x)$ is positive definite. Suppose $r = 1$, for $z \in D_{m-1}$, let $x = x_z + x^z$ be the orthogonal decomposition with respect to the line z, *i.e.* $x_z \in z$ and $x^z \perp z$. Let $R(x, z) = -(x_z, x_z)$ that is nonnegative since z is negative definite, and $R(x, z) = 0$ if and only if $x = 0$ or $z \in D_x$. Explicitly, let $x = (x_1, \ldots, x_m) \in V'$, $z = (z_1, \ldots, z_{m-1}) \in D_{m-1}$. Then

$$R(x, z) = \frac{(x_1\overline{z_1} + \cdots + x_{m-1}\overline{z_{m-1}} - x_m)\,(\overline{x_1}z_1 + \cdots + \overline{x_{m-1}}z_{m-1} - \overline{x_m})}{1 - z\overline{z}},$$

where we recall that $z\overline{z} = z_1\overline{z_1} + \cdots + z_{m-1}\overline{z_{m-1}}$. We define

$$\xi(x, z) = -\operatorname{Ei}(-2\pi R(x, z)).$$

For each nonzero element $x \in V'$, $\xi(x, \bullet)$ is a smooth function on $D_{m-1} - D_x$, and has logarithmic growth along D_x if D_x is not empty. Therefore, we can view it as a current $[\xi(x)] = [\xi(x, \bullet)]$ on D_{m-1}. Let $\varphi \in \left(\mathcal{S}(V'^r) \otimes A^{r,r}(D_{m-1})\right)^{\mathrm{U}(V')}$ $(1 \leq r \leq m-1)$ be the *Kudla–Millson form* constructed in [KM1986], and let

$$\omega(x) = \omega(x, \bullet) = \exp(2\pi \operatorname{tr} T(x))\varphi(x, \bullet).$$

We have the following proposition.

Proposition 4.2.1. *Let $x \in V'$ be a nonzero elements. Then we have*

$$\mathrm{dd}^c[\xi(x)] + \delta_{D_x} = [\omega(x)]$$

as currents on D_{m-1}.

We will only give a proof for $m = 2$, and the proof for general m is similar but involves tedious computations.

Proof. We start from showing that $\mathrm{dd}^c\xi(x) = \omega(x)$ holds away from D_x. Let $x = (x_1, x_2)$ and $z \in D_1 - D_x$. Sometimes we simply write R instead of $R(x)$ for short. Then we have the formula

$$\mathrm{dd}^c\xi(x) = \frac{1}{2\pi i}\left(\frac{\exp(-2\pi R)}{R^2}\left(R\partial\overline{\partial}R - \partial R \wedge \overline{\partial}R\right) - 2\pi\frac{\exp(-2\pi R)}{R}\partial R \wedge \overline{\partial}R\right). \tag{4.28}$$

Computing each term, we have

$$\begin{aligned}
R(x,z) &= \frac{(x_1\overline{z} - x_2)(\overline{x_1}z - \overline{x_2})}{1 - z\overline{z}};\\
\overline{\partial}R &= \frac{x_1(\overline{x_1}z - \overline{x_2})(1 - z\overline{z}) + (x_1\overline{z} - x_2)(\overline{x_1}z - \overline{x_2})z}{(1 - z\overline{z})^2}\mathrm{d}\overline{z};\\
\partial R &= \frac{\overline{x_1}(x_1\overline{z} - x_2)(1 - z\overline{z}) + (x_1\overline{z} - x_2)(\overline{x_1}z - \overline{x_2})\overline{z}}{(1 - z\overline{z})^2}\mathrm{d}z;\\
\partial\overline{\partial}R &= \left(\frac{x_1\overline{x_1}}{1 - z\overline{z}} + \frac{x_1\overline{z}(\overline{x_1}z - \overline{x_2}) + (x_1\overline{z} - x_2)(2\overline{x_1}z - \overline{x_2}) + 2Rz\overline{z}}{(1 - z\overline{z})^2}\right)\mathrm{d}z \wedge \mathrm{d}\overline{z};\\
\partial R \wedge \overline{\partial}R &= \left(\frac{x_1\overline{x_1}R}{1 - z\overline{z}} + \frac{\overline{x_1}z(x_1\overline{z} - x_2)R + x_1\overline{z}(\overline{x_1}z - \overline{x_2})R + R^2z\overline{z}}{(1 - z\overline{z})^2}\right)\mathrm{d}z \wedge \mathrm{d}\overline{z}.
\end{aligned}$$

Therefore,

$$R\partial\overline{\partial}R - \partial R \wedge \overline{\partial}R = \left(\frac{(x_1\overline{z} - x_2)(\overline{x_1}z - \overline{x_2})R}{(1 - z\overline{z})^2} - \frac{R^2z\overline{z}}{(1 - z\overline{z})^2}\right)\mathrm{d}z \wedge \mathrm{d}\overline{z} = R^2\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}; \tag{4.29}$$

and

$$\begin{aligned}
\partial R \wedge \overline{\partial}R &= \left(x_1\overline{x_1}(1 - z\overline{z}) + \overline{x_1}z(x_1\overline{z} - x_2) + x_1\overline{z}(\overline{x_1}z - \overline{x_2}) + Rz\overline{z}\right)\frac{R\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}\\
&= \left(x_1\overline{x_1} + \overline{x_1}zx_2 + x_1\overline{z}\overline{x_2} + x_1\overline{x_1}z\overline{z}\right)\frac{R\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}\\
&= \left((x,x) + (\overline{x_1}z - \overline{x_2})(x_1\overline{z} - x_2) + Rz\overline{z}\right)\frac{R\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}\\
&= \left(R(x,z) + (x,x)\right)\frac{R\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}. \qquad (4.30)
\end{aligned}$$

Plugging (4.29) and (4.30), we have that

$$(4.28) = (1 - 2\pi(R(x,z) + (x,x)))\exp(-2\pi R(x,z))\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{2\pi i(1 - z\overline{z})^2} = \omega(x,z).$$

The remaining discussion is same as in the proof of [Kud1997, Proposition 11.1], from Lemma 11.2 on Page 606. We will not repeat the detail. $\square$

The above proposition says that $\xi(x)$ is a Green function of logarithmic type for D_x. Now we consider $x = (x_1, \ldots, x_r) \in V'^r$ with nonsingular moment matrix $T(x)$. Using the star product of Green currents, we have a Green current

$$\Xi_x = [\xi(x_1)] * \cdots * [\xi(x_r)]$$

for D_x. As currents of degree (r, r), we have

$$\mathrm{dd}^c\left([\xi(x_1)] * \cdots * [\xi(x_r)]\right) + \delta_{D_x} = [\omega(x_1) \wedge \cdots \wedge \omega(x_r)] = [\omega(x)].$$

Definition 4.2.2 (Height functions (on $\mathcal{D}$))**.** For $x = (x_1, \ldots, x_m) \in V'^m$ with nonsingular moment matrix $T(x)$, we define the *height function (on $\mathcal{D}$)* to be

$$H(x)_\infty = \langle 1, \Xi_x \rangle = \langle 1, [\xi(x_1)] * \cdots * [\xi(x_m)] \rangle.$$

Since $\xi(hx, hz) = \xi(x, z)$ for $h \in \mathrm{U}(V')$, the height function satisfies $H(hx)_\infty = H(x)_\infty$, and thus depends only on the (nonsingular) moment matrix $T(x)$. Sometimes we simply write $H(T)_\infty$ for this function.

The following proposition claims that $H(T)_\infty$ is in fact invariant under the conjugation action of $\mathrm{U}(m)_\mathbb{R}$.

Proposition 4.2.3. *The height function $H(T)_\infty$ depends only on the eigenvalues of T. In other words, for every $k \in \mathrm{U}(m)_\mathbb{R}$, $H(kT\,{}^t\overline{k})_\infty = H(T)_\infty$.*

Proof. We prove by induction on m. The case $m = 2$ in left to the next subsection. Suppose that $m \geq 3$ and the proposition holds for $m - 1$. Since $\mathrm{U}(m)_\mathbb{R}$ is generated by diagonal matrices, permutation matrices, and the matrices of form

$$\begin{pmatrix} k' & \\ & 1 \end{pmatrix}, \qquad k' \in \mathrm{U}(m-1)_\mathbb{R},$$

We only need to prove that $H((x'k', x_m))_\infty = H((x', x_m))_\infty$, where $x = (x', x_m) \in V'^{m-1} \oplus V' = V'^m$

with $T(x) = T$. By definition,

$$\begin{aligned} H((x', x_m))_\infty &= \langle 1, [\xi(x_1)] * \cdots * [\xi(x_{m-1})]|_{D_{x_m}}\rangle + \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m) \\ &= H(x')_\infty + \int_{D_{m-1}} \omega(x') \wedge \xi(x_m). \end{aligned}$$

By induction, $H(x'k')_\infty = H(x')_\infty$ and by [KM1986, Theorem 3.2 (ii)], $\omega(x') = \omega(x'k')$. Therefore, $H((x'k', x_m))_\infty = H((x', x_m))_\infty$. □

4.2.2 Invariance under U(2): an exercise in Calculus

We consider the case $m = 2$. Suppose

$$T = \begin{pmatrix} d_1 & m \\ \overline{m} & d_2 \end{pmatrix}$$

with $d_1, d_2 \in \mathbb{R}$ and $m \in \mathbb{C}$. Choose a complex number ϵ with norm 1 such that $\epsilon^2 m \in \mathbb{R}$. Then

$$\begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} d_1 & m \\ \overline{m} & d_2 \end{pmatrix} {}^t\overline{\begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix}} = \begin{pmatrix} d_1 & \epsilon^2 m \\ \epsilon^2 m & d_2 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{R}).$$

We write elements of SO(2) in the form

$$k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R},$$

and write $T[\theta] = k_\theta T \,{}^t\overline{k_\theta} = k_\theta T \,{}^t k_\theta$. Since $\xi(\epsilon x) = \xi(x)$ for every $x \in V'$, to finish the proof of Proposition 4.2.3, we need only to prove the following proposition.

Proposition 4.2.4. *For every $T \in \mathrm{Sym}_2(\mathbb{R})$ with $\operatorname{sign} T = (1, 1)$, we have $H(T[\theta])_\infty = H(T)_\infty$.*

Suppose $T = \operatorname{diag}[a, -b]$ with $a, b > 0$,

$$T[\theta] = \begin{pmatrix} d_{1,\theta} & m_\theta \\ m_\theta & d_{2,\theta} \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{R}).$$

Let $x_0 = (\sqrt{2a}, 0) \in V'$, $y_0 = (0, \sqrt{2b}) \in V'$. For $\theta \in \mathbb{R}$, let

$$x_\theta = x_0 k_\theta = (x_{1,\theta}, x_{2,\theta}) = \cos\theta \cdot x_0 - \sin\theta \cdot y_0;$$
$$y_\theta = y_0 k_\theta = (y_{1,\theta}, y_{2,\theta}) = \sin\theta \cdot x_0 + \cos\theta \cdot y_0.$$

We have

$$\frac{\mathrm{d}x_\theta}{\mathrm{d}\theta} = -y_\theta; \qquad \frac{\mathrm{d}y_\theta}{\mathrm{d}\theta} = x_\theta; \qquad H(T[\theta])_\infty = H((x_\theta, y_\theta))_\infty.$$

Let

$$z_{x,\theta} = \frac{x_{2,\theta}}{x_{1,\theta}}; \qquad z_{y,\theta} = \frac{y_{2,\theta}}{y_{1,\theta}}.$$

Then

$$D_{x_\theta} = [z_{x,\theta}, 1]; \qquad D_{y_\theta} = [z_{y,\theta}, 1]$$

if not empty. In what follows, we adopt the convention that if $|z| \geq 1$, $f(z) = 0$ for any function f defined on a subset of $\mathbb{C}$. We record the following lemma, which appears as [Kud1997, Lemma 11.4].

Lemma 4.2.5. *We have that*

$$\begin{aligned} H((x_\theta, y_\theta))_\infty &= \xi(x_\theta, z_{y,\theta}) + \int_{D_1} \xi(y_\theta)\omega(x_\theta) \\ &= \xi(y_\theta, z_{x,\theta}) + \int_{D_1} \xi(x_\theta)\omega(y_\theta) \\ &= \xi(x_\theta, z_{y,\theta}) + \xi(y_\theta, z_{x,\theta}) - \int_{D_1} \mathrm{d}\xi(x_\theta) \wedge \mathrm{d}^c\xi(y_\theta). \end{aligned}$$

We consider the last integral above in general. Write $x = (x_1, x_2) \in V'$, $y = (y_1, y_2) \in V'$, $R_1 = R(x)$, and $R_2 = R(y)$. Define

$$\begin{aligned} I(T) = I((x, y)) &= -\int_{D_1} \mathrm{d}\xi(x) \wedge \mathrm{d}^c\xi(y) \\ &= -\frac{1}{4\pi i}\int_{D_1} \left(\partial + \overline{\partial}\right)\xi(x) \wedge \left(\partial - \overline{\partial}\right)\xi(y) \\ &= -\frac{i}{4\pi}\int_{D_1} \left(\partial\xi(x) \wedge \overline{\partial}\xi(y) + \partial\xi(y) \wedge \overline{\partial}\xi(x)\right) \\ &= -\frac{i}{4\pi}\int_{D_1} \frac{\exp\left(-2\pi(R_1 + R_2)\right)}{R_1 R_2}\left(\partial R_1 \wedge \overline{\partial}R_2 + \partial R_2 \wedge \overline{\partial}R_1\right). \end{aligned}$$

For $z \in D_1$, let

$$x(z) = (1 - z\overline{z})^{-1/2}(z, 1) \in V'; \qquad M = (x, x(z))\overline{(y, x(z))}.$$

We have the following lemma.

Lemma 4.2.6. *Let* $2m = (x, y)$*. Then*

$$\partial R_1 \wedge \overline{\partial} R_2 + \partial R_2 \wedge \overline{\partial} R_1 = 2\left(R_1 R_2 + \overline{m} M + m \overline{M}\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2};$$
$$\partial R_1 \wedge \overline{\partial} R_2 - \partial R_2 \wedge \overline{\partial} R_1 = 2\left(\overline{m} M - m \overline{M}\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}.$$

Proof. By definition,

$$R_1 = \frac{(x_1 \overline{z} - x_2)\left(\overline{x_1} z - \overline{x_2}\right)}{1 - z\overline{z}}.$$

Therefore,

$$\partial R_1 = \frac{(x_1 \overline{z} - x_2)\,\overline{x_1} + \overline{z} R_1}{1 - z\overline{z}} \mathrm{d}z,$$

and similarly for ∂R_2, $\overline{\partial} R_1$, and $\overline{\partial} R_2$. We compute that

$$\begin{aligned}
&\partial R_1 \wedge \overline{\partial} R_2 \\
&= \left((x_1 \overline{z} - x_2)\left(\overline{y_1} z - \overline{y_2}\right) \overline{x_1} y_1 + \left(\overline{y_1} z - \overline{y_2}\right) y_1 \overline{z} R_1 + (x_1 \overline{z} - x_2)\, \overline{x_1} z R_2 + z\overline{z} R_1 R_2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2} \\
&= \left((x_1 \overline{z} - x_2)\left(\overline{y_1} z - \overline{y_2}\right) \overline{x_1} y_1 + \left(\overline{y_1} z - \overline{y_2}\right) y_2 R_1 + (x_1 \overline{z} - x_2)\, \overline{x_1} z R_2 + R_1 R_2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2} \\
&= \left((x_1 \overline{z} - x_2)\left(\overline{y_1} z - \overline{y_2}\right)\left(\overline{x_1} y_1 + \frac{y_2\left(\overline{x_1} z - \overline{x_2}\right)}{1 - z\overline{z}}\right) + (x_1 \overline{z} - x_2)\, \overline{x_1} z R_2 + R_1 R_2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2} \\
&= \left((x_1 \overline{z} - x_2)\left(\overline{y_1} z - \overline{y_2}\right) \frac{\overline{x_1} y_1 - \overline{x_2} y_2 - \overline{x_1} z\left(y_1 \overline{z} - y_2\right)}{1 - z\overline{z}} + (x_1 \overline{z} - x_2)\, \overline{x_1} z R_2 + R_1 R_2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2} \\
&= \left(2\overline{m} M + R_1 R_2\right) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - z\overline{z})^2}.
\end{aligned}$$

The lemma follows from a similar calculation for $\partial R_2 \wedge \overline{\partial} R_1$. □

We define a morphism

$$\alpha : \mathbb{R} \times D_1 \to \mathrm{Her}_2(\mathbb{C})^{\det = 0} = \{h \in \mathrm{Her}_2(\mathbb{C}) \mid \det h = 0\}$$

of 3-dimensional real analytic spaces by the formula

$$\alpha(\theta, z) = \begin{pmatrix} R_1 & M \\ \overline{M} & R_2 \end{pmatrix} = \begin{pmatrix} (x_\theta, x(z))\overline{(x_\theta, x(z))} & (x_\theta, x(z))\overline{(y_\theta, x(z))} \\ \overline{(x_\theta, x(z))}(y_\theta, x(z)) & (y_\theta, x(z))\overline{(y_\theta, x(z))} \end{pmatrix},$$

and set $\alpha_\theta = \alpha(\theta, \bullet)$. By an easy computation, we see that

$$\frac{\mathrm{d}R_1}{\mathrm{d}\theta} = -\left(M + \overline{M}\right); \qquad \frac{\mathrm{d}R_2}{\mathrm{d}\theta} = M + \overline{M}; \qquad \frac{\mathrm{d}M}{\mathrm{d}\theta} = R_1 + R_2. \tag{4.31}$$

Therefore, $R_1 + R_2$ and $M - \overline{M}$ are independent of θ, which are the values at $\theta = 0$, respectively. In other words,

$$R_1 + R_2 = \frac{2az\overline{z} + 2b}{1 - z\overline{z}} = -2a + \frac{2(a+b)}{1 - z\overline{z}}; \tag{4.32}$$

$$M - \overline{M} = \frac{2\sqrt{ab}\,(z - \overline{z})}{1 - z\overline{z}}. \tag{4.33}$$

By Lemma 4.2.6, and the fact that $2m_\theta = (x_\theta, y_\theta) \in \mathbb{R}$, we have

$$\begin{aligned} I(T[\theta]) &= -\frac{i}{2\pi}\int_{D_1} \frac{\exp(-2\pi(R_1+R_2))}{R_1R_2}\left(R_1R_2 + m\left(M + \overline{M}\right)\right)\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(1-z\overline{z})^2} \\ &= I'(T[\theta]) + I''(T[\theta]), \end{aligned} \tag{4.34}$$

where

$$\begin{aligned} I'(T[\theta]) &= -\frac{i}{2\pi}\int_{D_1} \exp(-2\pi(R_1+R_2))\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(1-z\overline{z})^2}; \\ I''(T[\theta]) &= -\frac{i}{2\pi}\int_{D_1} \frac{\exp(-2\pi(R_1+R_2))}{R_1R_2} m\left(M + \overline{M}\right)\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(1-z\overline{z})^2}. \end{aligned}$$

By (4.32) and (4.33), the integral $I'(T[\theta])$ is independent of θ. There we only need to consider the second term $I''(T[\theta])$. Define a differential form of degree two on (the smooth locus of) $\mathrm{Her}_2(\mathbb{C})^{\det=0}$ as follows.

$$\Xi = -\frac{i}{4\pi}\frac{\exp(-2\pi(R_1+R_2))}{R_1R_2}\frac{M+\overline{M}}{M-\overline{M}}\mathrm{d}R_1\wedge\mathrm{d}R_2,$$

which has singularities along the locus $R_1R_2\left(M - \overline{M}\right) = 0$. We have the following lemma.

Lemma 4.2.7. *We have*

1. *For a fixed $\theta \in \mathbb{R}$,*

$$\alpha_\theta^*(\Xi) = -\frac{i}{2\pi}\frac{\exp(-2\pi(R_1+R_2))}{R_1R_2} m\left(M + \overline{M}\right)\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(1-z\overline{z})^2};$$

2. On $\mathrm{Her}_2(\mathbb{C})^{\det=0}$, *we have*

$$\mathrm{d}\Xi = \frac{i}{\pi}\frac{\exp(-2\pi(R_1+R_2))}{\left(M-\overline{M}\right)^2\left(M+\overline{M}\right)}\mathrm{d}\left(M-\overline{M}\right)\wedge \mathrm{d}R_1\wedge\mathrm{d}R_2.$$

Proof. 1. It follows from Lemma 4.2.6.

2. By the equality

$$\left(M+\overline{M}\right)^2-\left(M-\overline{M}\right)^2=4R_1R_2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\left(M-\overline{M}\right)}\frac{M+\overline{M}}{M-\overline{M}} = -\frac{4R_1R_2}{\left(M-\overline{M}\right)^2\left(M+\overline{M}\right)}.$$

Then it follows.

□

Let $D_1^+ = \{z\in D_1 \mid \operatorname{Im} z\geq 0\}$. Since $\alpha_\theta^*(\Xi)/dz\wedge d\overline{z}$ is invariant under $z\mapsto\overline{z}$, by Lemmas 4.2.5, 4.2.7 (1), and (4.34), we have

$$\begin{aligned}
&H(T[\theta_1])_\infty - H(T[\theta_0])_\infty\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+I(T[\theta])-I(T[\theta_0])\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+I''(T[\theta])-I''(T[\theta_0])\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+\int_{D_1}\alpha_{\theta_1}^*(\Xi)-\int_{D_1}\alpha_{\theta_0}^*(\Xi)\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+2\int_{D_1^+}\alpha_{\theta_1}^*(\Xi)-2\int_{D_1^+}\alpha_{\theta_0}^*(\Xi). \qquad (4.35)
\end{aligned}$$

We see that the form $\alpha_\theta^*(\Xi)$ has (possible) singularities when $R_1R_2=0$, *i.e.* at the (possible) points $z_{x,\theta}$, $z_{y,\theta}$. An easy calculation shows that

$$z_{x,\theta}=\frac{x_{2,\theta}}{x_{1,\theta}}=-\tan\theta\cdot\sqrt{\frac{b}{a}}\in\mathbb{R};\qquad z_{y,\theta}=\frac{y_{2,\theta}}{y_{1,\theta}}=\cot\theta\cdot\sqrt{\frac{b}{a}}\in\mathbb{R}.$$

We now assume that $0<\theta_0\leq\theta_1<\pi/2$. Then $0\in D_1$ will not be a singular point for $\theta\in[\theta_0,\theta_1]$. We need to evaluate

$$\int_{D_1^+}\alpha_{\theta_0}^*(\Xi)-\int_{D_1^+}\alpha_{\theta_1}^*(\Xi).$$

For any $\epsilon>0$ small enough, let

- $B_{1,\epsilon}$ be the (oriented) path $\{z=r\exp(i\epsilon)\mid r\in[0,1)\}$ from $r=0$ to $r=1$;

- $B_{2,\epsilon}$ the path $\{z = r\exp(i(\pi-\epsilon)) \mid r \in [0,1)\}$ from $r=1$ to $r=0$; and

- $D_\epsilon \subset D_1^+$ the area containing points on or above the lines $B_{1,\epsilon}$ and $B_{2,\epsilon}$.

By our assumption, $\alpha_\theta^*(\Xi)$ is nonsingular on D_ϵ for every $\theta \in [\theta_0, \theta_1]$. By Stokes' Theorem and the fact that $\exp(-2\pi(R_1+R_2))$ decays rapidly as $|z|$ goes to 1, we have

$$\int_{[\theta_0,\theta_1]\times D_\epsilon} \alpha^*(\mathrm{d}\Xi) = \int_{D_\epsilon} \alpha_{\theta_1}^*(\Xi) - \int_{D_\epsilon} \alpha_{\theta_0}^*(\Xi) + \int_{[\theta_0,\theta_1]\times(B_{1,\epsilon}\cup B_{2,\epsilon})} \alpha^*(\Xi). \tag{4.36}$$

Lemma 4.2.8. *We have*

$$\int_{[\theta_0,\theta_1]\times D_\epsilon} \alpha^*(\mathrm{d}\Xi) = 0.$$

Proof. By (4.31), (4.32) and (4.33), we have

$$\mathrm{d}R_1 = \partial R_1 + \overline{\partial} R_1 - \left(M + \overline{M}\right)\mathrm{d}\theta;$$

$$\mathrm{d}R_2 = \partial R_2 + \overline{\partial} R_2 + \left(M + \overline{M}\right)\mathrm{d}\theta;$$

$$\mathrm{d}\left(M - \overline{M}\right) = 2\sqrt{ab}\left(\partial\frac{z-\overline{z}}{1-z\overline{z}} + \overline{\partial}\frac{z-\overline{z}}{1-z\overline{z}}\right).$$

Therefore,

$$\begin{aligned}
&\alpha^*\left(\mathrm{d}\left(M-\overline{M}\right)\wedge \mathrm{d}R_1 \wedge \mathrm{d}R_2\right)\\
&= 2\sqrt{ab}\left(M+\overline{M}\right)\left(\partial\frac{z-\overline{z}}{1-z\overline{z}} \wedge \overline{\partial}(R_1+R_2) - \partial(R_1+R_2)\wedge\overline{\partial}\frac{z-\overline{z}}{1-z\overline{z}}\right)\\
&= 4\sqrt{ab}(a+b)\left(M+\overline{M}\right)\left(\partial\frac{z-\overline{z}}{1-z\overline{z}}\wedge\overline{\partial}\frac{1}{1-z\overline{z}} - \partial\frac{1}{1-z\overline{z}}\wedge\overline{\partial}\frac{z-\overline{z}}{1-z\overline{z}}\right).
\end{aligned}$$

Then by Lemma 4.2.7 (2), we have

$$\begin{aligned}
\alpha^*\left(\mathrm{d}\Xi\right) &= \frac{4i\sqrt{ab}(a+b)}{\pi}\frac{\exp(-2\pi(R_1+R_2))}{\left(M-\overline{M}\right)^2}\left(\partial\frac{z-\overline{z}}{1-z\overline{z}}\wedge\overline{\partial}\frac{1}{1-z\overline{z}} - \partial\frac{1}{1-z\overline{z}}\wedge\overline{\partial}\frac{z-\overline{z}}{1-z\overline{z}}\right)\\
&= \frac{4i\sqrt{ab}(a+b)}{\pi}\frac{\exp(-2\pi(R_1+R_2))}{\left(M-\overline{M}\right)^2}\frac{z+\overline{z}}{(1-z\overline{z})^3}\mathrm{d}z\wedge\mathrm{d}\overline{z}.
\end{aligned}$$

Since $z \mapsto -\overline{z}$ stabilizes the domain $[\theta_0,\theta_1]\times D_\epsilon$, and maps $\alpha^*\left(\mathrm{d}\Xi\right)/\mathrm{d}z\wedge\mathrm{d}\overline{z}$ to its negative, the integral is zero. $\square$

By the above lemma and (4.36), we have

$$\int_{D_1^+} \alpha_{\theta_0}^*(\Xi) - \int_{D_1^+} \alpha_{\theta_1}^*(\Xi) = \lim_{\epsilon\to 0}\int_{D_\epsilon} \alpha_{\theta_0}^*(\Xi) - \lim_{\epsilon\to 0}\int_{D_\epsilon} \alpha_{\theta_1}^*(\Xi) = \lim_{\epsilon\to 0}\int_{[\theta_0,\theta_1]\times(B_{1,\epsilon}\cup B_{2,\epsilon})} \alpha^*(\Xi). \tag{4.37}$$

A simple computation shows that on $[\theta_0, \theta_1] \times (B_{2,\epsilon} \cup B_{1,\epsilon})$, we have

$$\begin{aligned}
\alpha^*(\Xi) &= \frac{-i(a+b)}{\pi}\frac{\exp(-2\pi(R_1+R_2))}{R_1R_2}\frac{\left(M+\overline{M}\right)^2}{M-\overline{M}}\frac{r}{(1-r^2)^2}\mathrm{d}r\wedge\mathrm{d}\theta \\
&= \frac{-i(a+b)}{\pi}\frac{\exp(-2\pi(R_1+R_2))\left(M-\overline{M}\right)}{R_1R_2}\frac{r}{(1-r^2)^2}\mathrm{d}r\wedge\mathrm{d}\theta \\
&\quad + \frac{-4i(a+b)}{\pi}\frac{\exp(-2\pi(R_1+R_2))}{M-\overline{M}}\frac{r}{(1-r^2)^2}\mathrm{d}r\wedge\mathrm{d}\theta.
\end{aligned}$$

Since the integrations of the second term on two paths cancel each other, we have

$$\begin{aligned}
(4.37) &= \int_{\theta_0}^{\theta_1}\mathrm{d}\theta\frac{-i(a+b)}{\pi}\lim_{\epsilon\to 0}\int_{B_{1,\epsilon}\cup B_{2,\epsilon}}\frac{\exp(-2\pi(R_1+R_2))\left(M-\overline{M}\right)}{R_1R_2}\frac{r}{(1-r^2)^2}\mathrm{d}r \\
&= \int_{\theta_0}^{\theta_1}\mathrm{d}\theta\frac{4\sqrt{ab}(a+b)}{\pi}\lim_{\epsilon\to 0}\sin\epsilon\int_{B_{1,\epsilon}\cup B_{2,\epsilon}}\frac{\exp(-2\pi(R_1+R_2))}{R_1R_2}\frac{r^2}{(1-r^2)^3}\mathrm{d}r. \qquad (4.38)
\end{aligned}$$

To proceed, we need the following lemma.

Lemma 4.2.9. *Let $f(r)$ be a smooth function on $[0,1)$ that is rapidly decreasing as $r \to 1$. Then for any $c_1, c_2, d_1, d_2 > 0$,*

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\int_0^1 \frac{\sin\epsilon}{\left(c_1^2r^2+c_2^2-2c_1c_2r\cos\epsilon\right)\left(d_1^2r^2+d_2^2+2d_1d_2r\cos\epsilon\right)}f(r)\mathrm{d}r \\
&= \begin{cases} \frac{\pi c_1}{c_2(c_1d_2+c_2d_1)^2}f\left(\frac{c_2}{c_1}\right) & c_1 > c_2; \\ 0 & c_1 \le c_2. \end{cases}
\end{aligned}$$

Proof. The case $c_1 \le c_2$ follows from the assumption that f is rapidly decreasing. For the first case, we only need to prove that

$$\lim_{\epsilon\to 0^+}\int_0^1 \frac{\sin\epsilon}{c_1^2r^2+c_2^2-2c_1c_2r\cos\epsilon}\mathrm{d}r = \frac{\pi}{c_1c_2}. \tag{4.39}$$

The integral of the left-hand side of (4.39) (for small $\epsilon > 0$) equals

$$
\begin{aligned}
&\sin\epsilon \int_0^1 \frac{1}{\left(c_1 r - c_2\cos\epsilon\right)^2 + c_2^2\left(1-\cos\epsilon\right)} \\
&= \frac{\sin\epsilon}{c_1 c_2\sqrt{1-\cos\epsilon}} \int_{-\frac{\cos\epsilon}{\sqrt{1-\cos\epsilon}}}^{\frac{c_1 - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}} \frac{1}{\left(\frac{c_1 r - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}\right)^2 + 1} \mathrm{d}\left(\frac{c_1 r - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}\right) \\
&= \frac{\sin\epsilon}{c_1 c_2\sqrt{1-\cos\epsilon}} \left(\arctan\frac{c_1 r - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}} + \arctan\frac{\cos\epsilon}{\sqrt{1-\cos\epsilon}}\right).
\end{aligned}
$$

Let $\epsilon \to 0^+$, the limit is π/c_1c_2. $\square$

Applying the above lemma, we have that

$$
(4.38) = \int_{\theta_0}^{\theta_1} \sqrt{ab}(a+b)\left(\frac{\exp\left(-2\pi R_1(z_{y,\theta})\right)}{R_1(z_{y,\theta})}\frac{y_{1,\theta}y_{2,\theta}}{d_{2,\theta}^2} + \frac{\exp\left(-2\pi R_2(z_{x,\theta})\right)}{R_2(z_{x,\theta})}\frac{x_{1,\theta}x_{2,\theta}}{d_{1,\theta}^2}\right)\mathrm{d}\theta. \tag{4.40}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\mathrm{d}R_1(z_{y,\theta})}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(R_1\left(z_{y,\theta}\right) + R_2\left(z_{y,\theta}\right)\right) = \frac{4(a+b)r}{(1-r^2)^2}\,\Big|_{r=\frac{y_{2,\theta}}{y_{1,\theta}}} \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{y_{2,\theta}}{y_{1,\theta}}\right) = 2\sqrt{ab}(a+b)\frac{y_{1,\theta}y_{2,\theta}}{d_{2,\theta}^2}; \\
&\frac{\mathrm{d}R_2(z_{x,\theta})}{\mathrm{d}\theta} = 2\sqrt{ab}(a+b)\frac{x_{1,\theta}x_{2,\theta}}{d_{1,\theta}^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(4.40) &= \frac{1}{2}\left(\int_{R_1(z_{y,\theta_0})}^{R_1(z_{y,\theta_1})} \frac{\exp\left(-2\pi R_1(z_{y,\theta})\right)}{R_1(z_{y,\theta})}\mathrm{d}R_1(z_{y,\theta}) + \int_{R_2(z_{x,\theta_0})}^{R_2(z_{x,\theta_1})} \frac{\exp\left(-2\pi R_2(z_{x,\theta})\right)}{R_2(z_{x,\theta})}\mathrm{d}R_2(z_{x,\theta})\right) \\
&= \frac{1}{2}\left(\xi(x_{\theta_1}, z_{y,\theta_1}) + \xi(y_{\theta_1}, z_{x,\theta_1}) - \xi(x_{\theta_0}, z_{y,\theta_0}) - \xi(y_{\theta_0}, z_{x,\theta_0})\right),
\end{aligned}
$$

which, by (4.35), implies that

$$
H(T[\theta_1])_\infty - H(T[\theta_0])_\infty = 0 \tag{4.41}
$$

for $0 < \theta_0 \le \theta_1 < \pi/2$. Same argument works for intervals $(\pi/2, \pi)$, $(\pi, 3\pi/2)$ and $(3\pi/2, 2\pi)$, other than $(0, \pi/2)$. The constancy of $H(T[\theta])_\infty$ for all $\theta \in \mathbb{R}$ then follows from (4.41) and the continuity. This finishes the proof of Proposition 4.2.4.

4.3 An archimedean local Siegel–Weil formula

In this section, we set up a relation between derivatives of Whittaker integrals and the height functions defined in the previous section. Furthermore, we prove a local arithmetic analogue of the Siegel–Weil formula at an archimedean place for arbitrary dimensions.

4.3.1 Comparison on the hermitian domain

We propose and prove the following theorem, which we call the *archimedean local arithmetic Siegel–Weil formula*.

Theorem 4.3.1. *Let $T \in \mathrm{Her}_m(\mathbb{C})$ such that $\operatorname{sign} T = (m-1,1)$. Then we have*

$$W'_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} \exp(-2\pi \operatorname{tr} T) H(T)_\infty,$$

where $H(T)_\infty$ is defined in Definition 4.2.2.

By Proposition 4.2.3, we only need to prove for $T = \operatorname{diag}[a_1, \dots, a_{m-1}, -b]$ with $a_1, \dots, a_{m-1}, b \in \mathbb{R}_{>0}$. We let $x_j = (\dots, \sqrt{2a_j}, \dots) \in \mathbb{C}^m \cong V'$ with the j-th entry $\sqrt{2a_j}$ and all others zero for $j = 1, \dots, m-1$, and $x_m = (0, \dots, 0, \sqrt{2b})$. Then $H(T)_\infty = H((x_1, \dots, x_m))_\infty$. Since $(x_m, x_m) < 0$, we have $D_{x_m} = \emptyset$, and

$$H(T)_\infty = \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m).$$

Proof of Theorem 4.3.1. By Proposition 4.1.8, we need to prove that

$$\begin{aligned}
&(2\pi i)^{m-1} \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m) \\
= &\int_{D_{m-1}} \exp\left(-4\pi \left(a_1 w_1 \overline{w_1} + \cdots + a_{m-1} w_{m-1} \overline{w_{m-1}}\right)\right) \\
&\sum_{1 \le s_1 < \cdots < s_t \le m-1} (-4\pi)^t (m-1-t)! (a_{s_1} \cdots a_{s_t}) \left(1 + w_{s_1} \overline{w_{s_1}} + \cdots + w_{s_t} \overline{w_{s_t}}\right) \\
&(-\operatorname{Ei})\left(-4\pi b \left(1 + {}^t\overline{w} w\right)\right) (1 - z\overline{z})^{-m}\, \Omega.
\end{aligned} \tag{4.42}$$

By definition and (4.24), we have

$$R_j(z) := R(x_j, z) = \frac{2a_j z_j \overline{z_j}}{1 - z\overline{z}} = 2a_j w_j \overline{w_j}, \qquad j = 1, \ldots, m-1;$$
$$R_m(z) := R(x_m, z) = \frac{-2b}{1 - z\overline{z}} = -2b\left(1 + {}^t\overline{w}w\right).$$

Therefore, $\xi(x_m) = -\operatorname{Ei}\left(-4\pi b\left(1 + {}^t\overline{w}w\right)\right)$. The next step is to find an explicit formula for $\omega(x_j)$. By (4.28), we need to calculate $\overline{\partial}R_j$, ∂R_j, and $\partial\overline{\partial}R_j$ for $j = 1, \ldots, m-1$. Since $(1 - z\overline{z})\, R_j = 2a_j z_j \overline{z_j}$,

$$\overline{\partial}\left(1 - z\overline{z}\right) R_j + \left(1 - z\overline{z}\right) \overline{\partial}R_j = 2a_j z_j \mathrm{d}\overline{z_j}; \tag{4.43}$$

$$\overline{\partial}R_j = \frac{2a_j z_j \mathrm{d}\overline{z_j} + R_j \overline{\partial}\left(z\overline{z}\right)}{1 - z\overline{z}}. \tag{4.44}$$

Similarly,

$$\partial R_j = \frac{2a_j \overline{z_j} \mathrm{d}z_j + R_j \partial\left(z\overline{z}\right)}{1 - z\overline{z}}. \tag{4.45}$$

Differentiating (4.43) and plugging (4.44) and (4.45), we have

$$\partial\overline{\partial}\left(1 - z\overline{z}\right) R_j + \partial R_j \overline{\partial}\left(1 - z\overline{z}\right) + \partial\left(1 - z\overline{z}\right) \overline{\partial}R_j + \left(1 - z\overline{z}\right) \partial\overline{\partial}R_j = 2a_j \mathrm{d}z_j \mathrm{d}\overline{z_j},$$

which implies that

$$R_j = \frac{2a_j\left(1 - z\overline{z}\right) \mathrm{d}z_j \mathrm{d}\overline{z_j} + 2a_j \overline{z_j} \mathrm{d}z_j \overline{\partial}\left(z\overline{z}\right) + 2a_j z_j \partial\left(z\overline{z}\right) \mathrm{d}\overline{z_j} + 2R_j \partial\left(z\overline{z}\right) \overline{\partial}\left(z\overline{z}\right) + R_j\left(1 - z\overline{z}\right) \partial\overline{\partial}\left(z\overline{z}\right)}{\left(1 - z\overline{z}\right)^2}. \tag{4.46}$$

Taking wedge of (4.44) and (4.45), we have

$$\partial R_j \wedge \overline{\partial}R_j = \frac{4a_j^2 z_j \overline{z_j} \mathrm{d}z_j \mathrm{d}\overline{z_j} + 2a_j R_j \overline{z_j} \mathrm{d}z_j \overline{\partial}\left(z\overline{z}\right) + 2a_j R_j z_j \partial\left(z\overline{z}\right) \mathrm{d}\overline{z_j} + R_j^2 \partial\left(z\overline{z}\right) \overline{\partial}\left(z\overline{z}\right)}{\left(1 - z\overline{z}\right)^2}. \tag{4.47}$$

Combining (4.46) and (4.47), we have

$$\frac{1}{R_j^2}\left(R_j \partial\overline{\partial}R_j - \partial R_j \wedge \overline{\partial}R_j\right) = \frac{\partial\left(z\overline{z}\right) \overline{\partial}\left(z\overline{z}\right)}{\left(1 - z\overline{z}\right)^2} + \frac{\partial\overline{\partial}\left(z\overline{z}\right)}{1 - z\overline{z}}. \tag{4.48}$$

For simplicity, we make some substitutions. Let

$$\omega = \partial\left(z\overline{z}\right)\overline{\partial}\left(z\overline{z}\right) + \left(1 - z\overline{z}\right)\partial\overline{\partial}\left(z\overline{z}\right);$$

$$\omega_j = \left(1 - z\overline{z}\right)\overline{z_j}\mathrm{d}z_j\mathrm{d}\overline{z_j} + \overline{z_j}\mathrm{d}z_j\overline{\partial}\left(z\overline{z}\right) + z_j\partial\left(z\overline{z}\right)\mathrm{d}\overline{z_j} + w_j\overline{w_j}\partial\left(z\overline{z}\right)\overline{\partial}\left(z\overline{z}\right), \qquad j = 1, \cdots, m-1.$$

Then we have

$$2\pi i\omega(x_j) = -\partial\overline{\partial}\xi(x_j) = \exp\left(-4\pi a_j w_j\overline{w_j}\right)\left(\omega - 4\pi a_j\omega_j\right)\left(1 - z\overline{z}\right)^2.$$

Therefore, to prove (4.42), we only need to prove the following equality of $(m-1, m-1)$-forms on D_{m-1},

$$\bigwedge_{j=1}^{m-1}\left(\omega - 4\pi a_j\omega_j\right) = \sum_{s_1<\cdots<s_t}(-4\pi)^t(m-1-t)!(a_{s_1}\cdots a_{s_t})\left(1 + w_{s_1}\overline{w_{s_1}} + \cdots + w_{s_t}\overline{w_{s_t}}\right)\left(1 - z\overline{z}\right)^{m-2}\Omega,$$

which follows from the claim that for every subset $\{s_1 < \cdots < s_t\} \subset \{1, \ldots, m-1\}$, we have

$$\omega_{s_1} \wedge \cdots \wedge \omega_{s_t} \wedge \omega^{m-1-t} = (m-1-t)!\left(1 + w_{s_1}\overline{w_{s_1}} + \cdots + w_{s_t}\overline{w_{s_t}}\right)(1 - z\overline{z})^{m-2}\,\Omega.$$

This will be proved in the next lemma where, without lost of generality, we assume that $s_j = j$. The theorem then follows. $\square$

Lemma 4.3.2. *Let w_j, Ω, ω and ω_j be as above. For any integer $0 \le t \le m-1$, we have the following equality of $(m-1, m-1)$-forms on D_{m-1}*

$$\left(\bigwedge_{j=1}^{t}\omega_j\right)\bigwedge\omega^{m-1-t} = (m-1-t)!\left(1 + \sum_{j=1}^{t}w_j\overline{w_j}\right)\left(1 - z\overline{z}\right)^{m-2}\Omega.$$

The proof will occupy the next subsection.

4.3.2 Proof of Lemma 4.3.2

For $j = 1, \ldots, m-1$, we let

$$\sigma_j = \overline{z_j}\mathrm{d}z_j\overline{\partial}\left(z\overline{z}\right); \qquad \sigma_j' = z_j\partial\left(z\overline{z}\right)\mathrm{d}\overline{z_j}; \qquad \delta_j = \left(1 - z\overline{z}\right)\mathrm{d}z_j\mathrm{d}\overline{z_j}.$$

Then

$$\sum_{k=1}^{m-1} \sigma_k = \sum_{k=1}^{m-1} \sigma'_k; \qquad \omega = \sum_{k=1}^{m-1} \left(\sigma_k + \delta_k\right); \qquad \omega_j = \delta_j + \sigma_j + \sigma'_j + w_j \overline{w_j} \sum_{k=1}^{m-1} \sigma_k;$$

and

$$\sigma_j \wedge \sigma_j = 0; \qquad \sigma'_j \wedge \sigma'_j = 0; \qquad \delta_j \wedge \delta_j = 0.$$

Introduce the following $(m-1) \times (m-1)$ matrix

$$Z = \begin{pmatrix} \overline{z_1} z_1 & \overline{z_2} z_1 & \cdots & \overline{z_{m-1}} z_1 \\ \overline{z_1} z_2 & \overline{z_2} z_2 & \cdots & \overline{z_{m-1}} z_2 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{z_1} z_{m-1} & \overline{z_2} z_{m-1} & \cdots & \overline{z_{m-1}} z_{m-1} \end{pmatrix}.$$

Recall the notation $Z_{J,K}$ as in the proof of Lemma 4.1.7 for subsets $J, K \subset \{1, \ldots, m-1\}$ with $|J| = |K|$. It is easy to see that $|Z_{J,K}| \neq 0$ only if $|J| \leq 1$, where in the later case, $\left|Z_{\{j\},\{k\}}\right| = \overline{z_j} z_k$ and $\left|Z_{\emptyset,\emptyset}\right| = 1$.

Consider three subsets $I, J, K \subset \{1, \ldots, m-1\}$ with $|I| + |J| + |K| = m - 1$. Write

$$\sigma_I = \bigwedge_{i \in I} \sigma_i; \qquad \sigma'_J = \bigwedge_{j \in J} \sigma'_j; \qquad \delta_K = \bigwedge_{k \in K} \delta_k.$$

We have the following equality

$$\sigma_I \sigma'_J \delta_K := \sigma_I \wedge \sigma'_J \wedge \delta_K = \begin{cases} \epsilon_{I,J,K} \left| Z_{I, \overline{J \cup K}} \right| \left| Z_{\overline{I \cup K}, J} \right| (1 - z\overline{z})^{|K|} \, \Omega & (I \cup J) \cap K = \emptyset; \\ 0 & (I \cup J) \cap K \neq \emptyset. \end{cases}$$

Here, $\epsilon_{I,J,K} \in \{-1, 0, 1\}$ is a factor depending only on I, J, K. It is nonzero only if $|I| \leq 1$ and $|J| \leq 1$. Explicitly,

$$\sigma_I \sigma'_J \delta_K = \begin{cases} \overline{z_i} z_i \overline{z_j} z_j \left(1 - z\overline{z}\right)^{m-3} \Omega & i \neq j, I = \{i\}, J = \{j\}, K = \overline{I \cup J}; \\ -\overline{z_i} z_j \overline{z_j} z_i \left(1 - z\overline{z}\right)^{m-3} \Omega & i \neq j, I = J = \{i\}, K = \overline{I \cup \{j\}}; \\ \overline{z_i} z_i \left(1 - z\overline{z}\right)^{m-2} \Omega & I \cup J = \{i\}, K = \overline{\{i\}}; \\ \left(1 - z\overline{z}\right)^{m-1} \Omega & I = J = \emptyset, K = \{1, \ldots, m-1\}. \end{cases}$$

For a subset P of $\{1,\ldots,m\}$, we set $w_P = \prod_{p\in P} w_p$ and $\overline{w_P} = \prod_{p\in P}\overline{w_p}$. Then

$$\begin{aligned}
\left(\bigwedge_{j=1}^{t}\omega_j\right)\bigwedge\omega^{m-1-t} &= \bigwedge_{j=1}^{t}\left(\delta_j+\sigma_j+\sigma_j'+w_j\overline{w_j}\sum_{k=1}^{m-1}\sigma_k\right)\bigwedge\left(\sum_{k=1}^{m-1}\sigma_k+\sum_{k=1}^{m-1}\delta_k\right)^{m-1-t}\\
&= \left(\sum_{L\coprod M\coprod N\coprod P=\{1,\ldots,t\}}\delta_L\sigma_M\sigma_N' w_P\overline{w_P}\left(\sum_{k=1}^{m-1}\sigma_k\right)^{|P|}\right)\\
&\quad\left(\sum_{\substack{Q\subset 1,\ldots,m-1\\|Q|\le m-1-t}}\frac{(m-1-t)!}{(m-1-t-|Q|)!}\left(\sum_{k=1}^{m-1}\sigma_k\right)^{m-1-t-|Q|}\delta_Q\right)\\
&= \sum_{L\coprod M\coprod N\coprod P=\{1,\ldots,t\}}\ \sum_{\substack{Q\subset 1,\ldots,m-1\\|Q|\le m-1-t}} T_{L,M,N,P,Q},
\end{aligned}\tag{4.49}$$

where

$$T_{L,M,N,P,Q} = \frac{(m-1-t)!}{(m-1-t-|Q|)!}\delta_{L\cup Q}\sigma_M\sigma_N' w_P\overline{w_P}\left(\sum_{k=1}^{m-1}\sigma_k\right)^{|P|+m-1-t-|Q|}.$$

It is easy to see that if $T_{L,M,N,P,Q}\neq 0$, then $|Q|\geq m-2-t$. We enumerate all cases where $T_{L,M,N,P,Q}$ may be nonzero.

Case I: $|Q| = m-1-t$. Then $|P|\leq 1$:

Case I-1: $|P| = 0$. Then $Q=\{t+1,\ldots,m-1\}$ and $|M|\leq 1$, $|N|\leq 1$:

Case I-1a: $M=\{m\}$ and $N=\{n\}$ for $m\neq n\in\{1,\cdots,t\}$. Then the sum of corresponding terms is

$$\sum T_{L,M,N,P,Q} = (m-1-t)!\sum_{\substack{m,n=1\\m\neq n}}^{t} z_m\overline{z_m}z_n\overline{z_n}\,(1-z\overline{z})^{m-3}\,\Omega. \tag{4.50}$$

Case I-1b: $M\cup N=\{m\}$ for $1\leq m\leq t$. Then the sum of corresponding terms is

$$\sum T_{L,M,N,P,Q} = 2(m-1-t)!\sum_{m=1}^{t} z_m\overline{z_m}\,(1-z\overline{z})^{m-2}\,\Omega. \tag{4.51}$$

Case I-1c: $M=N=\emptyset$. Then the corresponding term is

$$T_{L,M,N,P,Q} = T_{\{1,\ldots,t\},\emptyset,\emptyset,\emptyset,\{t+1,\ldots,m-1\}} = (m-1-t)!\,(1-z\overline{z})^{m-1}\,\Omega. \tag{4.52}$$

Case I-2: $|P| = 1$. Then $M=N=\emptyset$. Suppose $P=\{p\}$ for $1\leq p\leq t$. Then $Q=\{p,t+1,\ldots,m-1\}-\{q\}$ for some q inside $\{p,t+1,\ldots,m-1\}$. The sum of the corresponding terms is

$$\sum T_{L,M,N,P,Q} = (m-1-t)! \sum_{p=1}^{t} w_p \overline{w_p} \left(z_p \overline{z_p} + \sum_{q=t+1}^{m-1} z_q \overline{z_q} \right) (1 - z\overline{z})^{m-2} \Omega. \tag{4.53}$$

Case II: $|Q| = m - 2 - t$. Then $M = N = P = \emptyset$ and $|Q| = \{t+1, \ldots, m-1\} - \{q\}$ for some q inside $\{t+1, \ldots, m-1\}$. The sum of the corresponding terms is

$$\sum T_{L,M,N,P,Q} = (m-1-t)! \sum_{q=t+1}^{m-1} z_q \overline{z_q} (1 - z\overline{z})^{m-2} \Omega. \tag{4.54}$$

Taking the sum from (4.50) to (4.54), we have

$$\begin{aligned}
(4.49) &= (m-1-t)! (1 - z\overline{z})^{m-2} \Omega \\
&\left(\sum_{p=1}^{t} w_p \overline{w_p} \left(z_p \overline{z_p} + \sum_{q=t+1}^{m-1} z_q \overline{z_q} \right) + \frac{\sum_{\substack{m,n=1 \\ m \neq n}}^{t} z_m \overline{z_m} z_n \overline{z_n}}{1 - z\overline{z}} + 2 \sum_{m=1}^{t} z_m \overline{z_m} + \sum_{q=t+1}^{m-1} z_q \overline{z_q} + (1 - z\overline{z}) \right) \\
&= (m-1-t)! (1 - z\overline{z})^{m-2} \Omega \\
&\left(1 + \sum_{m=1}^{t} z_m \overline{z_m} + \frac{\sum_{\substack{m,n=1 \\ m \neq n}}^{t} z_m \overline{z_m} z_n \overline{z_n} + \sum_{p=1}^{t} z_p \overline{z_p} \left(z_p \overline{z_p} + \sum_{q=t+1}^{m-1} z_q \overline{z_q} \right)}{1 - z\overline{z}} \right) \\
&= (m-1-t)! (1 - z\overline{z})^{m-2} \Omega \frac{\sum_{m=1}^{t} z_m \overline{z_m}}{1 - z\overline{z}} \\
&= (m-1-t)! \left(1 + \sum_{j=1}^{t} w_j \overline{w_j} \right) (1 - z\overline{z})^{m-2} \Omega.
\end{aligned}$$

Lemma 4.3.2 is proved.

4.3.3 Comparison on Shimura varieties

We use previous results to compute the archimedean local height pairing on the unitary Shimura varieties with respect to suitable Green currents. We recall some notations from 3.1.1 and 3.1.2. Let $n \geq 1$ be an integer. We have a totally positive definition incoherent hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank $2n$, and $\mathbb{H} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathrm{U}(\mathbb{V})$. For (sufficiently small) open compact subgroup K of $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})$, we have the Shimura variety $\mathrm{Sh}_K := \mathrm{Sh}_K(\mathbb{H})$. Suppose that $K = \prod_{\mathfrak{p} \in \Sigma_{\mathrm{fin}}} K_{\mathfrak{p}}$ is decomposable. Let $\phi_\alpha = \phi_\infty^0 \otimes \phi_{\alpha,\mathrm{fin}}$ $(\alpha = 1, 2)$ be decomposable Schwartz functions with $\phi_{\alpha,\mathrm{fin}} \in \mathcal{S}(\mathbb{V}_{\mathrm{fin}}^n)^K$. Suppose that $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}} \in \mathcal{S}(\mathbb{V}_{\mathfrak{p}}^{2n})_{\mathrm{reg}}$ for some finite place $\mathfrak{p}$ of F. Then the generating series $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$ do not meet on Sh_K providing $g_\alpha \in P'_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$.

Definition 4.3.3 (Volume of open compact)**.** For every finite place $\mathfrak{p}$ of F, we define a measure $\mathrm{d}h$ on $\mathbb{H}(\mathbb{A}_{\mathrm{fin}})$, which depends only on the additive character ψ_{fin}, as follows. By [Ral1987, Lemma 4.2], there is a unique Haar measure $\mathrm{d}'h_{\mathfrak{p}}$ on $\mathrm{U}(\mathbb{V}_{\mathfrak{p}})$ such that for every nonsingular matrix $T \in \mathrm{Her}_{2n}(E_{\mathfrak{p}})$ such that $\mathbf{O}_T$ is nonempty, and $\Phi \in \mathcal{S}(\mathbb{V}_{\mathfrak{p}}^{2n})$,

$$W_T(0, e, \Phi) = \gamma_{\mathbb{V}_{\mathfrak{p}}} b_{2n,\mathfrak{p}}(0)^{-1} \int_{\mathrm{U}(\mathbb{V}_{\mathfrak{p}})} \Phi(h_v^{-1} x_T) \mathrm{d}'h_v,$$

where x_T is any element in $\mathbf{O}_T$, and $b_{2n,\mathfrak{p}}$ is defined in (2.7). By Lemma 2.3.6, for almost all $\mathfrak{p}$, the volume of K_v with respect to $\mathrm{d}'h_{\mathfrak{p}}$ is 1. We define

$$\mathrm{d}h = \frac{1}{2b_{2n}(0)} \prod_{\mathfrak{p} \in \Sigma_{\mathrm{fin}}} \mathrm{d}'h_{\mathfrak{p}},$$

where $b_{2n} = \prod_{v \in \Sigma} b_{2n,v}$ is (a product of) global Tate L-factors. Let $\mathrm{Vol}(K)$ be the volume of K with respect to the measure $\mathrm{d}h$.

In what follows, we fix an archimedean place ι of F, and $\iota' \in \{\iota^\circ, \iota^\bullet\}$. Let $V = V^{(\iota)}$, $H = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V)$, and $\mathcal{D} = \mathcal{D}^{(\iota')}$. Assume that there exists a finite place $\mathfrak{p}$ of F such that $\phi_{\mathfrak{p}}(0) = 0$. Let $\mathrm{Her}_n^+(E)$ be the subset of $\mathrm{Her}_n(E)$ consisting of totally positive definite matrices. For every $T \in \mathrm{Her}_n^+(E)$, we choose an element $x_T \in V^n$ such that $T(x) = T$. Then for $g \in P'_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$, we have the generating series

$$Z_\phi(g) = \sum_{T \in \mathrm{Her}_n^+(E)} \sum_{h \in H_{x_T}(\mathbb{A}_{\mathrm{fin}}) \backslash H(\mathbb{A}_{\mathrm{fin}})/K} \left(\omega_\chi(g)\phi\right)(T, h^{-1}x_T) Z(h^{-1}x_T)_K,$$

where $H_{x_T} \subset H$ is the stabilizer of x_T. Recall that under the uniformization

$$(\mathrm{Sh}_K)_{\iota'}^{\mathrm{an}} \cong H(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_{\mathrm{fin}})/K,$$

the special cycle $Z(h^{-1}x_T)_K$ is represented by the points $(z, h'h)$, where $z \perp V_{x_T}$ and h' fixes all elements in V_{x_T}. In other words, if we identify $\mathcal{D}$ with D_{2n-1}, then z is in D_{x_T}. Choose a set of representatives $h_1, \dots, h_l$ of the double coset $H(\mathbb{Q}) \backslash H(\mathbb{A}_{\mathrm{fin}})/K$ such that h_1 is the identity element. Let $[Z_\phi(g)_{\iota'}]_{h_1}^{\mathrm{an}}$ be the restriction of (the ι'-analytification of) $Z_\phi(g)$ to the neutral component. Then it is the image of

$$\sum_{x \in V^n, T(x) \in \mathrm{Her}_n^+(E)} \left(\omega_\chi(g)\phi\right)(T(x), x) D_x$$

under the projection map $\mathcal{D} \to (H(\mathbb{Q}) \cap K) \backslash \mathcal{D}$. Write $g_\iota = n(b)m(a)[k_1, k_2]$ under the Iwasawa decomposition as in 4.1.1. Then

$$\Xi_\phi(g)_{\iota',h_1} = \sum_{x\in V^n, T(x)\in \mathrm{Her}_n^+(E)} \left(\omega_\chi(g)\phi\right)(T(x),x)\Xi_{xa} \tag{4.55}$$

projects to a current on $(H(\mathbb{Q})\cap K)\backslash \mathcal{D}$, which is a Green current for $[Z_\phi(g)_{\iota'}]^{\mathrm{an}}_{h_1}$. By Hecke translation under h_i $(i=2,\dots,l)$, we obtain the a Green current $\Xi_\phi(g)_{\iota'}$ for $Z_\phi(g)_{\iota'}$. The following is our main theorem of this chapter.

Theorem 4.3.4. *Let ι, ι' be as above. Let $\phi_\alpha = \phi^0_\infty \otimes \phi_{\alpha,\mathrm{fin}}$ $(\alpha=1,2)$ be decomposable Schwartz functions with $\phi_{\alpha,\mathrm{fin}} \in \mathcal{S}(\mathbb{V}^n_{\mathrm{fin}})^K$. Suppose that $\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}} \in \mathcal{S}(\mathbb{V}^{2n}_{\mathfrak{p}})_{\mathrm{reg}}$ for some finite place $\mathfrak{p}$ of F. Then for $g_\alpha \in P'_{\mathfrak{p}}H'(\mathbb{A}^{\mathfrak{p}}_F)$,*

$$E_\iota(0, \imath(g_1,g_2^\vee), \phi_1\otimes\phi_2) = -2\,\mathrm{Vol}(K)\langle \left(Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota'}\right), \left(Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota'}\right)\rangle_{\mathrm{Sh}_K},$$

where E_ι is defined as (2.15); and the right-hand side is the local height pairing on Sh_K at the place ι'.

Proof. It is clear that we can assume $g_{\alpha,\iota} = m(a_\alpha)$ with $a_\alpha \in \mathrm{GL}_n(E_{\iota'})$ for $\alpha = 1,2$. Then we have

$$\begin{aligned}
&\langle \left(Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota'}\right), \left(Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota'}\right)\rangle_{\mathrm{Sh}_K} \\
&= \sum_{i=1}^{l} \int_{(H(\mathbb{Q})\cap K)\backslash\mathcal{D}} \left(\Xi_{\omega_\chi(h_i)\phi_1}(g_1)_{\iota',h_1}\right) * \left(\Xi_{\omega_\chi(h_i)\phi_2}(g_2)_{\iota',h_1}\right) \\
&= \sum_{i=1}^{l} \int_{(H(\mathbb{Q})\cap K)\backslash\mathcal{D}} \left(\sum_{x_1\in V^n, T(x_1)\in\mathrm{Her}_n^+(E)} \left(\omega_\chi(g_1)\phi_1\right)(T(x_1), h_i^{-1}x_1)\Xi_{x_1a_1} \right) \\
&\qquad\qquad * \left(\sum_{x_2\in V^n, T(x_2)\in\mathrm{Her}_n^+(E)} \left(\omega_\chi(g_2)\phi_2\right)(T(x_2), h_i^{-1}x_2)\Xi_{x_2a_2} \right) \\
&= \sum_{i=1}^{l} \sum_{x_1\in V^n, T(x_1)\in\mathrm{Her}_n^+(E)} \sum_{x_2\in V^n, T(x_2)\in\mathrm{Her}_n^+(E)} \left(\omega''_\chi\left(\imath(g_1,g_2^\vee)\right)(\phi_1\otimes\phi_2)\right)(T((x_1,x_2)), h_i^{-1}(x_1,x_2)) \\
&\quad \int_{\mathcal{D}} \Xi_{x_1a_1} * \Xi_{x_2a_2}
\end{aligned} \tag{4.56}$$

Let

$$a = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}.$$

Then

$$(4.56) = \sum_{i=1}^{l} \sum_{x \in V^{2n}, T(x) \in \mathrm{Her}_{2n}^{+}(E)} \left(\omega_{\chi}''\left(\imath(g_1, g_2^{\vee})\right)(\phi_1 \otimes \phi_2)\right)(T(x), h_i^{-1}x) H({}^t\overline{a}T(x)a)_\infty$$

$$= \sum_{T} H({}^t\overline{a}Ta)_\infty \prod_{v \in \Sigma_\infty} \left(\omega_{\chi_v}''\left(\imath(g_{1,v}, g_{2,v}^{\vee})\right)\Phi_v^0\right)(T)$$

$$\prod_{\mathfrak{p} \in \Sigma_{\mathrm{fin}}} \sum_{h_\mathfrak{p} \in \mathrm{U}(\mathbb{V}_\mathfrak{p})/K_\mathfrak{p}} \left(\omega_{\chi_\mathfrak{p}}''\left(\imath(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^{\vee})\right)(\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}})\right)(h_\mathfrak{p}^{-1}x_T), \tag{4.57}$$

where the sum is taken over all $T \in \mathrm{Her}_{2n}^{+}(E)$ that is the moment matrix of some $x_T \in V^{2n}$. There are three cases.

Case I: $v = \iota$. By (4.3) and Theorem 4.3.1 for ${}^t\overline{a}Ta$, we have

$$H({}^t\overline{a}Ta)_\infty \left(\omega_{\chi_\iota}''\left(\imath(g_{1,\iota}, g_{2,\iota}^{\vee})\right)\Phi_\iota^0\right)(T) = \gamma_{\mathbb{V}_\iota}^{-1}\frac{\Gamma_{2n}(2n)}{(2\pi)^{4n^2}} W_T'(0, \imath(g_{1,\iota}, g_{2,\iota}^{\vee}), \Phi_\iota^0).$$

By (2.7), we have

$$H({}^t\overline{a}Ta)_\infty \left(\omega_{\chi_\iota}''\left(\imath(g_{1,\iota}, g_{2,\iota}^{\vee})\right)\Phi_\iota^0\right)(T) = \gamma_{\mathbb{V}_\iota}^{-1} b_{2n,\iota}(0) W_T'(0, \imath(g_{1,\iota}, g_{2,\iota}^{\vee}), \Phi_\iota^0). \tag{4.58}$$

Case II: $v \in \Sigma_\infty$ and $v \neq \iota$. By (4.3) and Proposition 4.1.5 (2), we have

$$\left(\omega_{\chi_v}''\left(\imath(g_{1,v}, g_{2,v}^{\vee})\right)\Phi_v^0\right)(T) = \gamma_{\mathbb{V}_v}^{-1} b_{2n,v}(0) W_T(0, \imath(g_{1,v}, g_{2,v}^{\vee}), \Phi_v^0). \tag{4.59}$$

Case III: $v \in \Sigma_{\mathrm{fin}}$. By Definition 4.3.3,

$$\sum_{h_\mathfrak{p} \in \mathrm{U}(\mathbb{V}_\mathfrak{p})/K_\mathfrak{p}} \left(\omega_{\chi_\mathfrak{p}}''\left(\imath(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^{\vee})\right)(\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}})\right)(h_\mathfrak{p}^{-1}x_T)$$

$$= \gamma_{\mathbb{V}_\mathfrak{p}}^{-1} b_{2n,\mathfrak{p}}(0) \left(\int_{K_\mathfrak{p}} \mathrm{d}'h_\mathfrak{p}\right)^{-1} W_T(0, \imath(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^{\vee}), \phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}) \tag{4.60}$$

After plugging (4.58), (4.59) and (4.60) in (4.57), the theorem follows. □

Chapter 5

Comparison at finite places: good reduction

We compare local terms of analytic and arithmetic kernel functions at an unramified finite place when $n = 1$. In 5.1, we introduce the integral models for the Shimura curves, and extend the special cycles and the generating series to the models. At a good place, we need to calculate certain intersection multiplicity on the smooth model. This is done in 5.2. The computation of the derivative of Whittaker integrals at an unramified place and the comparison of the analytic and arithmetic side are the content of 5.3.

We fix some notations for this chapter, which may differ from the previous ones. Let $F/\mathbb{Q}_p$ be a finite extension and E/F a quadratic extension of fields with $\mathrm{Gal}(E/F) = \{1, \tau\}$. We fix a uniformizer ϖ of F and let q be the cardinality of the residue field of F. Let $V^{\pm}$ be the 2-dimensional E-hermitian space with $\epsilon(V^{\pm}) = \pm 1$, which is unique up to isometry and $H^{\pm} = \mathrm{U}(V^{\pm})$. Let $\Lambda^{\pm}$ be a maximal $\mathcal{O}_E$-lattice in $V^{\pm}$, on which the hermitian form takes values in $\mathcal{O}_E$. Let $K_0^{\pm}$ be the stabilizer of $\Lambda^{\pm}$ in $H^{\pm}$, which is a maximal compact subgroup. Recall that we have local groups $H' = H_1$, $H'' \cong H_2$, P, etc.

Recall that we let ι_i $(i = 1, \dots, d)$ be all embeddings of F into $\mathbb{C}$ and ι_i°, $\iota_i^{\bullet}$ those of E above ι_i as in Notation 2.1.1. Let us fix more notations for Chapter 5 and 6.

- For any rational prime p, we fix an isomorphism $\iota_{(p)} : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ once and for all.
- For $\mathfrak{p}$ a finite place of F, let $\mathfrak{p}^{\circ}$ (resp. $\mathfrak{p}^{\circ}$, $\mathfrak{p}^{\bullet}$) be that (resp. those) of E over $\mathfrak{p}$ if $\mathfrak{p}$ is nonsplit (resp. split) in E. We fix a uniformizer ϖ of $F_{\mathfrak{p}}$.

- For a number field F, $T_F = \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{m,F}$ and $T^1_{F'} = \mathrm{Res}_{F/\mathbb{Q}}\,F'^{\times,1}$ for any quadratic extension F'/F, where $F'^{\times,1} = \ker(\mathrm{Nm} : F'^{\times} \to F^{\times})$. If F is totally real, let $F^+ \subset F$ be the subset consisting of all totally positive elements.

- For every finite extension $L/\mathbb{Q}_p$ of local fields with ring of integers $\mathcal{O}_L$ and maximal ideal $\mathfrak{q} \subset \mathcal{O}_L$ and $l \geq 0$, let U^l_L be the subgroup of $\mathcal{O}^{\times}_L$ consisting of elements that are congruent to 1 modulo $\mathfrak{q}^l$. Denote by L^0 the maximal unramified extension of L. Let L^l be the finite extension of L^0 corresponding to U^l_L through local class field theory. Finally, let $\widehat{L^l}$ be the completion of L^l, whose ring of integers is denoted by $\mathcal{O}_{\widehat{L^l}}$.

- We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.

5.1 Integral models

In the next four subsections, we will assume that $F \neq \mathbb{Q}$. The rest case is slightly different and in fact simpler, which will be summarized in the last subsection. We will adopt of Carayol [Car1986] to study the integral model of unitary Shimura curves and their special cycles.

5.1.1 Change of Shimura data

Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ $(1 \leq r \leq d)$ be all places of F dividing p, and $\mathfrak{p}^\circ$ a place of E above $\mathfrak{p}$. We assume that the embedding $\iota_{(p)} \circ \iota_1^\circ : E \hookrightarrow \mathbb{C}_p$ induces the place $\mathfrak{p}^\circ$. As before, we suppress ι_1° and ι_1 for the nearby objects. We have the hermitian space V over E of dimension 2 whose signature is $(1,1)$ at ι_1 and $(2,0)$ elsewhere, the unitary group H over $\mathbb{Q}$, and the Shimura curve $M_K = \mathrm{Sh}_K(H, X)$ for a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_{\mathrm{fin}})$, which is a smooth projective curve defined over $\iota_1^\circ(E)$. Recall that X is the conjugacy class of the Hodge map $h : \mathbb{S} \to H_{\mathbb{R}}$ defined by

$$z = x + iy \mapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \times z, 1, \ldots, 1 \right) \in H(\mathbb{R}) \subset \left(\mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{R}^\times} \mathbb{C}^\times\right) \times \left(\mathbb{H}^\times \times_{\mathbb{R}^\times} \mathbb{C}^\times\right)^{d-1},$$

where we identify $T_E(\mathbb{R})$ with $(\mathbb{C}^\times)^d$ through $(\iota_1^\circ, \ldots, \iota_d^\circ)$. We denote by $\nu : H \to T_E^1$ the determinant map. Then we have the 0-dimensional Shimura variety $L_K = \mathrm{Sh}_{\nu(K)}(T_E^1, \nu(X))$, and a smooth morphism, which is also denoted by, $\nu : M_K \to L_K$ of $\iota_1^\circ(E)$-schemes such that the fiber of each geometric point is connected.

Let us define a subgroup $K_{\mathfrak{p},n}$ of $\mathrm{U}(V_\mathfrak{p})$ for every integer $n \geq 0$. Since $V_\mathfrak{p}$ is (isometric to) either V^+ or V^- (resp. V^+), we have the lattice $\Lambda^\pm$ (resp. Λ^+) if $\mathfrak{p}$ is nonsplit (resp. split). We define $K_{\mathfrak{p},n}$ to be the subgroup of $K_0^\pm$ consisting of elements that have trivial action on $\Lambda^\pm / \varpi^n \Lambda^\pm$. Then $K_{\mathfrak{p},0} = K_0^\pm$ is a maximal compact subgroup. For $K = K_{\mathfrak{p},n} \times K^\mathfrak{p}$, we write $M_{n,K^\mathfrak{p}}$ (resp. $L_{n,K^\mathfrak{p}}$) for M_K (resp. L_K).

Notation 5.1.1. For simplicity, we introduce the following notation

$$H_{\mathrm{fin}}^\mathfrak{p} = \mathrm{U}(V \otimes_F \mathbb{A}_{F,\mathrm{fin}}^\mathfrak{p}).$$

Then $K^\mathfrak{p}$ is an open compact subgroup of $H_{\mathrm{fin}}^\mathfrak{p}$.

Since $F \neq \mathbb{Q}$, the Shimura datum (H, X) is not of PEL type. We need to change Shimura data in order to obtain the moduli interpretations and integral models. This is analogous to the situation in

[Car1986, YZZa, Zha2001a, Zha2001b], and we refer the detailed proof of various facts to [Car1986]. We choose a negative number $\lambda \in \mathbb{Q}$ such that the extension $\mathbb{Q}(\sqrt{\lambda})$ is split at p, and the CM extension $F^\dagger = F(\sqrt{\lambda})/F$ with $\mathrm{Gal}(F^\dagger/F) = \{1, \tau^\dagger\}$ is not isomorphic to E/F. We fix also a square root λ' of λ in $\mathbb{C}$ with positive imaginary part, and a square root λ_p of λ in $\mathbb{Q}_p$. Let ι_i^1 (resp. ι_i^2) be the embeddings of $F^\dagger$ into $\mathbb{C}$ above ι_i $(i = 1, \dots, d)$ which sends $\sqrt{\lambda}$ to λ' (resp. $-\lambda'$). Since p is split in $\mathbb{Q}(\sqrt{\lambda})$, $\mathfrak{p}_i$ for $i = 1, \dots, r$ are all split in $F^\dagger$. We denote by $\mathfrak{p}_i^1$ (resp. $\mathfrak{p}_i^2$) the place above $\mathfrak{p}_i$ which sends $\sqrt{\lambda}$ to λ_p (resp. $-\lambda_p$), and assume that $\iota_{(p)} \circ \iota_1^1$ induces $\mathfrak{p}_1^1$.

By the Hasse principle, there is a unique up to isometry quaternion algebra B over F, such that B, as an F-quadratic space (of dimension 4), is isometric to the underlying F-quadratic space of V, which is equipped with the quadratic form $\mathrm{Tr}_{E/F}(-,-)$, where $(-,-)$ is the hermitian form on V. More precisely, when v is finite, $B_v = B \otimes_F F_v$ is division if and only if v is nonsplit and $V_v \cong V^-$; and $B_{\iota_1}(\mathbb{R}) \cong \mathrm{Mat}_2(\mathbb{R})$, $B_{\iota_i}(\mathbb{R}) \cong \mathbb{H}$ for $i > 1$. We identify two quadratic spaces B and V through a fixed isometry and hence V has both left and right multiplication by B. We fix an embedding $E \hookrightarrow B$, through which the action of E on V induced from the left multiplication of B coincides with the E-vector space structure of V. Let $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$ with center $T \cong T_F$ and

$$G^\dagger = G \times_T T_{F^\dagger} \xrightarrow{\nu^\dagger} T \times T^1_{F^\dagger},$$

where

$$\nu^\dagger(g \times z) = \left(\mathrm{Nm}\, g \cdot z z^{\tau^\dagger}, \frac{z}{z^{\tau^\dagger}}\right).$$

Consider the subtorus $T^\dagger = \mathbb{G}_{m,\mathbb{Q}} \times T^1_{F^\dagger}$, and let $H^\dagger$ be the preimage of $T^\dagger$ under $\nu^\dagger$. Define the Hodge map $h^\dagger : \mathbb{S} \to H^\dagger_{\mathbb{R}} \subset G_{\mathbb{R}} \times_{T_{\mathbb{R}}} T_{F^\dagger,\mathbb{R}}$ by

$$z = x + iy \mapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \times 1, \mathbf{1}_2 \times z^{-1}, \dots, \mathbf{1}_2 \times z^{-1} \right), \tag{5.1}$$

and let $X^\dagger$ be the $H^\dagger(\mathbb{R})$-conjugacy class of $h^\dagger$, where we identify $T_{F^\dagger}(\mathbb{R})$ with $(\mathbb{C}^\times)^d$ through $(\iota_1^1, \dots, \iota_d^1)$. We then have the Shimura curve $M^\dagger_{K^\dagger} = \mathrm{Sh}_{K^\dagger}(H^\dagger, X^\dagger)$ that is defined over $\iota_1^1(F^\dagger)$ for an open compact subgroup $K^\dagger$ of $H^\dagger(\mathbb{A}_{\mathrm{fin}})$. Similarly, we have the smooth morphism

$$\nu^\dagger : M^\dagger_{K^\dagger} \to L^\dagger_{K^\dagger}.$$

Moreover, $h^\dagger(i)$ defines a complex structure on V_{ι_1}, and hence V_{ι_1} becomes a complex hermitian space of dimension 2 that is isometric to its original complex hermitian space structure inherited from the E-hermitian space V. In such a way, $X^\dagger$ can be identified with the set of negative definite complex lines in V_{ι_1}. Therefore, $X^\dagger$ is isomorphic to X as hermitian symmetric domains.

As in [Car1986, Section 2.2], $H^\dagger$ is a group of symplectic similitude. In fact, let $B^\dagger = B \otimes_F F^\dagger$, and $b \mapsto \bar{b}$ be the involution of the second kind on $B^\dagger$, which is the tensor product of the canonical involution on B and the conjugation on $F^\dagger$. Consider the underlying $\mathbb{Q}$-vector space $V^\dagger$ of $B^\dagger$. Define a symplectic form by

$$\psi^\dagger(v, w) = \mathrm{Tr}_{F^\dagger/\mathbb{Q}} \left(\sqrt{\lambda}\, \mathrm{Tr}_{B^\dagger/F^\dagger}(v\overline{w}) \right)$$

for $v, w \in B^\dagger$. Then $H^\dagger$ can be identified with the group of $B^\dagger$-linear symplectic similitude of $(V^\dagger, \psi^\dagger)$ through the left action $hv = v \cdot h^{-1}$. In particular, $H^\dagger(\mathbb{Q}_p)$ can be identified with the group $\mathbb{Q}_p^\times \times \prod_{i=1}^r B_{\mathfrak{p}_i}^\times$. For every open compact subgroups $K_p^{\dagger,\mathfrak{p}}$ of $\prod_{i=2}^r B_{\mathfrak{p}_i}^\times$, and $K^{\dagger,p}$ of $H^\dagger(\mathbb{A}_{\mathrm{fin}}^p)$, we simply write $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ for $M^\dagger_{K^\dagger}$, where $K^\dagger = \mathbb{Z}_p^\times \times \mathcal{O}_{B_\mathfrak{p}}^\times \times K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p}$, and similarly for $L^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$.

We let

$$M_{K;\mathfrak{p}^\circ} = M_K \times_E E_{\mathfrak{p}^\circ}, \qquad M^\dagger_{K^\dagger;\mathfrak{p}} = M^\dagger_{K^\dagger} \times_F F^\dagger_{\mathfrak{p}_1^1};$$
$$L_{K;\mathfrak{p}^\circ} = L_K \times_E E_{\mathfrak{p}^\circ}, \qquad L^\dagger_{K^\dagger;\mathfrak{p}} = L^\dagger_{K^\dagger} \times_F F^\dagger_{\mathfrak{p}_1^1},$$

where $F^\dagger_{\mathfrak{p}_1^1}$ is naturally a subfield of $E_{\mathfrak{p}^\circ}$, which is identified with $F_\mathfrak{p}$. Since H and $H^\dagger$ have the same derived subgroup, which is also the derived subgroup of G, we have the follow result of Carayol.

Proposition 5.1.2 (Section 4 of [Car1986]). *Let $K^\mathfrak{p} \subset H^\mathfrak{p}_{\mathrm{fin}}$ (Notation 5.1.1) be an open compact subgroup that is decomposable and sufficiently small. Then there is an open compact subgroup $K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p} \subset \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}^p_{\mathrm{fin}})$, such that the geometric neutral components $M^\circ_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$ and $M^{\dagger,\circ}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$ are defined and isomorphic over $E^0_{\mathfrak{p}^\circ}$.*

5.1.2 Moduli interpretations and integral models: minimal level

From the Hodge map $h^\dagger$ (5.1), we have a Hodge filtration

$$0 \subset \mathrm{Fil}^0(V^\dagger_\mathbb{C}) = (V^\dagger_\mathbb{C})^{0,-1} \subset V^\dagger_\mathbb{C}.$$

We define

$$t^\dagger(b) = \mathrm{tr}\left(b; V^\dagger_\mathbb{C}/\,\mathrm{Fil}^0(V^\dagger_\mathbb{C})\right) \in \iota_1^1(F^\dagger)$$

for $b \in B^\dagger$. For sufficiently small open compact subgroup $K^\dagger$, the curve $M^\dagger_{K^\dagger}$ represents the following moduli functor on the category of locally noetherian $\iota_1^1(F^\dagger)$-schemes (cf. [Kot1992]): for every such scheme S, $M^\dagger_{K^\dagger}(S)$ is the set of equivalence classes of quadruples $(A, \theta, i, \bar{\eta})$, where

- A is an abelian scheme over S of dimension $4d$;
- $\theta : A \to A^\vee$ is a polarization;
- $i : B^\dagger \hookrightarrow \mathrm{End}^0_S(A)$ is a monomorphism of $\mathbb{Q}$-algebras, such that $\mathrm{tr}(i(b); \mathrm{Lie}_S(A)) = t^\dagger(b)$ and $\theta \circ i(b) = i(\bar{b})^\vee \circ \theta$ for all $b \in B^\dagger$;
- $\bar{\eta}$ is a K-level structure, that is, for a chosen geometric point s on each connected component of S, $\bar{\eta}$ is a $\pi_1(S, s)$-invariant $K^\dagger$-orbit of $B^\dagger \otimes \mathbb{A}_{\mathrm{fin}}$-linear symplectic similitude $\eta : V^\dagger \otimes \mathbb{A}_{\mathrm{fin}} \to \mathrm{H}_1^{\text{ét}}(A_s, \mathbb{A}_{\mathrm{fin}})$, where the pairing on the latter space is the θ-Weil pairing.

Two quadruples $(A, \theta, i, \bar{\eta})$ and $(A', \theta', i', \bar{\eta}')$ are equivalent if there is an isogeny $A \to A'$ that sends θ to a $\mathbb{Q}^\times$-multiple of θ', i to i', and $\bar{\eta}$ to $\bar{\eta}'$.

Taking the base change by $\iota_{(p)}$, we obtain the functor $M^\dagger_{K^\dagger;\mathfrak{p}}$ over the completion $[\iota_1^1(F^\dagger)]^\wedge_{\iota_{(p)}} \cong F_\mathfrak{p}$. For every element $(A, \theta, i, \bar{\eta})$ in $M^\dagger_{K^\dagger;\mathfrak{p}}(S)$, $\mathrm{Lie}_S(A)$ is a $B_p^\dagger = B^\dagger \otimes_\mathbb{Q} \mathbb{Q}_p$-module. Since the algebra $B_p^\dagger = B \otimes_F (F^\dagger \otimes \mathbb{Q}_p)$ decomposes as

$$B_p^\dagger = B_1^1 \oplus B_2^1 \oplus \cdots \oplus B_r^1 \oplus B_1^2 \oplus B_2^2 \oplus \cdots \oplus B_r^2, \tag{5.2}$$

where $B_i^j = B^\dagger \otimes_F F^\dagger_{\mathfrak{p}_i^j}$ is isomorphic to $B_{\mathfrak{p}_i}$ as an $F_{\mathfrak{p}_i}$-algebra, the $B_p^\dagger$-module $\mathrm{Lie}_S(A)$ decomposes as

$$\mathrm{Lie}_S(A) = \left(\bigoplus_{i=1}^r \mathrm{Lie}_S(A)_i^1 \right) \bigoplus \left(\bigoplus_{i=1}^r \mathrm{Lie}_S(A)_i^2 \right),$$

and

$$A_{p^\infty} = \left(\bigoplus_{i=1}^r (A_{p^\infty})_i^1 \right) \bigoplus \left(\bigoplus_{i=1}^r (A_{p^\infty})_i^2 \right),$$

for the p-divisible group A_{p^∞} of A. Since the involution $b \mapsto \bar{b}$ on $B_p^\dagger$ interchanges the factors B_i^1 and B_i^2, by computing the trace, we see that the condition $\mathrm{tr}(i(b); \mathrm{Lie}_S(A)) = t^\dagger(b)$ is equivalent to the following:

$$\mathrm{tr}\left(b \in B_1^2; \mathrm{Lie}_S(A)_1^2\right) = \mathrm{Tr}_{B_1^2/F_\mathfrak{p}}(b); \qquad \mathrm{Lie}_S(A)_i^2 = 0, i \geq 2. \tag{5.3}$$

Fix a maximal order $\Lambda_i^2 = \mathcal{O}_{B_{\mathfrak{p}_i}}$ of B_i^2 for each $i = 1, \dots, r$, and let Λ_i^1 be the dual of Λ_i^2. Then

$$\Lambda_p = \left(\bigoplus_{i=1}^r \Lambda_i^1\right) \bigoplus \left(\bigoplus_{i=1}^r \Lambda_i^2\right) \subset \left(\bigoplus_{i=1}^r (V_p^\dagger)_i^1\right) \bigoplus \left(\bigoplus_{i=1}^r (V_p^\dagger)_i^2\right) = V_p^\dagger := V^\dagger \otimes \mathbb{Q}_p$$

is a $\mathbb{Z}_p$-lattice in $V_p^\dagger$ that is selfdual under $\psi^\dagger$. There is a unique maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}^\dagger \subset B^\dagger$ such that $\overline{\mathcal{O}^\dagger} = \mathcal{O}^\dagger$, and $\mathcal{O}^\dagger_{\mathfrak{p}_i^2} = \mathcal{O}_{B_{\mathfrak{p}_i}}$ that acts on Λ_i^2, where $\mathcal{O}^\dagger_{\mathfrak{p}_i^2}$ is the B_i^2-component of $\mathcal{O}^\dagger \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \subset B^\dagger \otimes_{\mathbb{Q}} \mathbb{Q}_p = B_p^\dagger$ under the decomposition (5.2). Then the moduli functor $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$ is isomorphic to the following one on the category of locally noetherian $F_{\mathfrak{p}}$-schemes: for every such scheme S, $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}(S)$ is the set of equivalence classes of quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ where

- A is an abelian scheme over S of dimension $4d$;
- $\theta : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}^\dagger \hookrightarrow \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}$ such that (5.3) is satisfied, and $\theta \circ i(b) = i(\bar{b})^\vee \circ \theta$ for all $b \in \mathcal{O}^\dagger$;
- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure, that is, a $\pi_1(S,s)$-invariant $K^{\dagger,p}$-orbit of $B^\dagger \otimes \mathbb{A}^p_{\mathrm{fin}}$-linear symplectic similitude $\eta^p : V^\dagger \otimes \mathbb{A}^p_{\mathrm{fin}} \to \mathrm{H}_1^{\text{ét}}(A_s, \mathbb{A}^p_{\mathrm{fin}})$;
- $\bar{\eta}_p^{\mathfrak{p}}$ is a $K_p^{\dagger,\mathfrak{p}}$-level structure, that is, a $\pi_1(S,s)$-invariant $K_p^{\dagger,\mathfrak{p}}$-orbit of isomorphisms of $\mathcal{O}^\dagger$-modules $\eta_p^{\mathfrak{p}} : \bigoplus_{i=2}^r \Lambda_i^2 \to \bigoplus_{i=2}^r \mathrm{H}_1^{\text{ét}}(A_s, \mathbb{Z}_p)_i^2$.

Two quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ and $(A', \theta', i', (\bar{\eta}^p)', (\bar{\eta}_p^{\mathfrak{p}})')$ are equivalent if there is a prime-to-p isogeny $A \to A'$ that carries θ to a $\mathbb{Z}_{(p)}^\times$-multiple of θ', i to i', $\bar{\eta}^p$ to $(\bar{\eta}^p)'$, and $\bar{\eta}_p^{\mathfrak{p}}$ to $(\bar{\eta}_p^{\mathfrak{p}})'$.

We will extend the previous moduli functor to the category of locally noetherian schemes over $\mathrm{Spec}\, \mathcal{O}_{F_{\mathfrak{p}}}$ to construct an integral model of $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$. Let us consider an abelian scheme (A, θ, i) that is a part of the datum defined above, but with the scheme S over $\mathcal{O}_{F_{\mathfrak{p}}}$. Through θ, we see that $(A_{p^\infty})_i^1$ and $(A_{p^\infty})_i^2$ are Cartier dual to each other. We replace (5.3) by the following condition:

$$\mathrm{tr}\left(b \in \mathcal{O}_{B_{\mathfrak{p}}} \subset B_1^2; \mathrm{Lie}_S(A)_1^2\right) = \mathrm{Tr}_{B_1^2/F_{\mathfrak{p}}}(b) \in \mathcal{O}_{F_{\mathfrak{p}}}; \qquad \mathrm{Lie}_S(A)_i^2 = 0, i \geq 2. \tag{5.4}$$

This means that the p-divisible group $(A_{p^\infty})_i^2$ is ind-étale for $i \geq 2$. We let $T_pA = \varprojlim_n A[p^n]$ that is a pro-scheme over S. It has an action by $\mathcal{O}^\dagger \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Then $T_pA(S)_i^2$ is isomorphic to Λ_i^2 as $\mathcal{O}^\dagger$-modules if S is connected and simply-connected.

We define a moduli functor $\underline{\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}}$ on the category of locally noetherian schemes over $\mathcal{O}_{F_{\mathfrak{p}}}$: for every such scheme S, $\underline{\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}}(S)$ is the set of equivalence classes of quintuple $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ where

- (A, θ, i) is as in the last moduli problem but satisfies (5.4);
- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure;
- $\bar{\eta}^{\mathfrak{p}}_p$ is a $\pi_1(S,s)$-invariant $K^{\dagger,\mathfrak{p}}_p$-orbit of isomorphisms of $\mathcal{O}^\dagger$-modules $\eta^{\mathfrak{p}}_p : \bigoplus_{i=2}^r \Lambda_i^2 \to \bigoplus_{i=2}^r T_p A(s)_i^2$.

Two quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p)$ and $(A', \theta', i', (\bar{\eta}^p)', (\bar{\eta}^{\mathfrak{p}}_p)')$ are equivalent if there exists a prime-to-p isogeny $A \to A'$ satisfying the same requirements in the last moduli problem. For sufficiently small $K^{\dagger,\mathfrak{p}}_p \times K^{\dagger,p}$, this moduli functor is represented by a regular scheme denoted by $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$, which is flat and projective over $\operatorname{Spec}\mathcal{O}_{F_\mathfrak{p}}$. By Proposition 5.1.2, we obtain a regular scheme $\mathcal{M}_{0,K^\mathfrak{p}}$ that is flat and projective over $\operatorname{Spec}\mathcal{O}_{E_{\mathfrak{p}^\circ}}$ whose generic fiber is (isomorphic to) $M_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$. Here, we also need to use the fact that $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ is stable for $K^{\dagger,p}$ small, and the results in [DM1969, Section 1] to make the descent argument. By construction, the neutral components of $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \times_{\mathcal{O}_{F_\mathfrak{p}}} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ and $\mathcal{M}_{0,K^\mathfrak{p}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ are isomorphic.

We denote by $(\mathcal{A}, \theta, i)$ (that is a part of the datum of) the universal object over $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$. We also denote by $\mathcal{X}^\dagger = (\mathcal{A}_{p^\infty})^2_1 \to \mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ the universal p-divisible group with the action by $\mathcal{O}_{B_\mathfrak{p}}$ and another action by $\prod_{i=2}^r B^\times_{\mathfrak{p}_i} \times H^\dagger(\mathbb{A}^p_{\mathrm{fin}})$ that is compatible with the action on the underlying scheme $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$. We have also a p-divisible group $\mathcal{X} \to \mathcal{M}_{0,K^\mathfrak{p}}$ with an action by $H^\mathfrak{p}_{\mathrm{fin}}$ that is compatible with the action on $\mathcal{M}_{0,K^\mathfrak{p}}$.

Remark 5.1.3. In fact, when $\mathfrak{p} \mid 2$ and $B_\mathfrak{p}$ is division, the condition (5.4) is not enough. One needs to impose that $(A_{p^\infty})^2_1$ is *special* (*cf.* [BC1991, Section II.2]) for geometric points of characteristic p.

Let us consider the case where $\epsilon(V_\mathfrak{p}) = 1$, that is, $V_\mathfrak{p}$ is isometric to V^+; $B_\mathfrak{p}$ is isomorphic to $\mathrm{Mat}_2(F_\mathfrak{p})$; or $\mathrm{U}(V_\mathfrak{p})$ is quasi-split. Before we proceed, we introduce some notations. Let R be a (commutative) ring (with a unit) and M a (left) R-module (or p-divisible group according to the context). Let $m > 0$ be an integer. We denote by $M^\sharp = M^m$ (arranged in a column) as a left $\mathrm{Mat}_m(R)$-module in the natural way. Conversely, for any left $\mathrm{Mat}_m(R)$-module N, we denote by $N^\flat = eN$ the (left) R-module, where $e = \mathrm{diag}[1, 0, \dots, 0] \in \mathrm{Mat}_m(R)$, and the action is given by $r.(en) = (e \times \mathrm{diag}[r, \dots, r]).n$ for $r \in R$ and $n \in N$. It is easy to see that the pair of adjoint functors $(-^\sharp, -^\flat)$ induce an equivalence between the category of left R-modules and the category of left $\mathrm{Mat}_m(R)$-modules.

We identify $\Lambda^2_1 = \mathcal{O}_{B_\mathfrak{p}}$ with $\mathrm{Mat}_2(\mathcal{O}_{F_\mathfrak{p}})$, and hence $\mathcal{O}^\times_{B_\mathfrak{p}}$ with $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$, respectively. By the above discussion, we can replace the first part of (5.4) by the following one:

$$\mathrm{tr}\left(b \in \mathcal{O}_{F_\mathfrak{p}}; \mathrm{Lie}_S(A)^{2,\flat}_1\right) = b, \tag{5.5}$$

in the moduli problem $\underline{\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}}$.

Consider a geometric point $s : \operatorname{Spec}\mathbb{F} \to \mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ of characteristic p and let $\widehat{\mathscr{O}_{(s)}}$ be the completion of the henselization of the local ring at s. By the Serre–Tate theorem, $\widehat{\mathscr{O}_{(s)}}$ is the universal deformation ring of $(\mathcal{A}_s, \theta_s, i_s)$, which is isomorphic to the one of $(\mathcal{A}_{s,p^\infty}, \theta_s, i_s)$. This is in turn isomorphic to the deformation ring of the p-divisible group $\mathcal{X}_s^{\dagger,\flat} = (\mathcal{A}_{s,p^\infty})_1^{2,\flat}$, which is an $\mathcal{O}_{F_\mathfrak{p}}$-module of dimension 1 and height 2. Therefore, $\widehat{\mathscr{O}_{(s)}}$ is isomorphic to $\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}[[t]]$. We have the following result of Carayol [Car1986, Section 6].

Proposition 5.1.4. *The scheme $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^\mathfrak{p}}$) is smooth and projective over $\mathcal{O}_{F_\mathfrak{p}}$ (resp. $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$).*

For a geometric point s of characteristic p of $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^\mathfrak{p}}$), there are two cases. We say s is *ordinary* if the formal part of $\mathcal{X}_s^\dagger$ (resp. $\mathcal{X}_s$) is of height 1; *supersingular* if $\mathcal{X}_s^\dagger$ (resp. $\mathcal{X}_s$) is formal. We denote by $[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathrm{ss}}$ (resp. $[\mathcal{M}_{0,K^\mathfrak{p}}]_{\mathrm{ss}}$) the supersingular locus of the scheme $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^\mathfrak{p}}$).

5.1.3 Basic abelian scheme

In order to obtain the moduli interpretation of special cycles, we will construct a special abelian scheme, which we name the *basic abelian scheme*. We fix an imaginary element μ in E, that is, an element $\mu \neq 0$ such that $\mu^\tau = -\mu$. Since we are interested only at the place $\mathfrak{p}$, we identify the following commutative diagram

$$\begin{tikzcd}
\iota_1^\circ(E) \arrow[rr, hook] & & \iota_{(p)}(\widehat{\iota_1^\circ(E)}) & \\
& \iota_1^1(F^\dagger) \arrow[rr, hook] & & \iota_{(p)}(\widehat{\iota_1^1(F^\dagger)}) \arrow[ul, hook] \\
\iota_1(F) \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & \iota_{(p)}(\widehat{\iota_1(F)}) \arrow[ur, "="] &
\end{tikzcd}$$

with

$$\begin{tikzcd}
E \arrow[rr, hook] & & E_{\mathfrak{p}^\circ} & \\
& F^\dagger \arrow[rr, hook] & & F_\mathfrak{p} \arrow[ul, hook] \\
F \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & F_\mathfrak{p}, \arrow[ur, "="] &
\end{tikzcd}$$

where the completion is taken inside $\mathbb{C}_p$.

Let $E^\dagger = E \otimes_F F^\dagger$ be a CM field of degree $4d$ that is a subalgebra of $B^\dagger$, which extends the fixed embedding $E \hookrightarrow B$. The involution on $B^\dagger$ induces $e \mapsto \overline{e}$ that fixes the maximal totally real subfield contained in $E^\dagger$. The maps

$$\iota_i^\circ \otimes \iota_i^j : E \otimes_F F^\dagger \to \mathbb{C} \otimes_\mathbb{R} \mathbb{C} \to \mathbb{C}; \quad \iota_i^\bullet \otimes \iota_i^j : E \otimes_F F^\dagger \to \mathbb{C} \otimes_\mathbb{R} \mathbb{C} \to \mathbb{C}$$

for $i = 1, \ldots, d$ and $j = 1, 2$ provide $4d$ different embeddings of $E^\dagger$ into $\mathbb{C}$, where $\mathbb{C} \otimes_\mathbb{R} \mathbb{C} \to \mathbb{C}$ is simply the multiplication map. We choose a CM type

$$\Phi = \{\iota_1^\circ \otimes \iota_1^1, \iota_1^\circ \otimes \iota_1^2; \iota_i^\circ \otimes \iota_i^1, \iota_i^\bullet \otimes \iota_i^1 \mid i = 2, \ldots, d\}$$

of $E^\dagger$. Then Φ determines a Hodge map $h^\ddagger : \mathbb{S} \to T^\ddagger_\mathbb{R}$, where $T^\ddagger$ is the subtorus of $\mathrm{Res}_{E^\dagger/\mathbb{Q}} \mathbb{G}_{m,E^\dagger}$ consisting of elements e such that $e\overline{e} \in \mathbb{G}_{m,\mathbb{Q}}$. We have the Shimura varieties $M^\ddagger_{K^\ddagger} = \mathrm{Sh}_{K^\ddagger}(T^\ddagger, \{h^\ddagger\})$ that parameterize abelian varieties over $E^{\dagger,\Phi}$ with Complex Multiplication by $E^\dagger$ of type Φ. It is finite and projective over $\mathrm{Spec}\, E^{\dagger,\Phi}$, where $E^{\dagger,\Phi}$ is the reflex field of $(E^\dagger, \Phi)$.

To be more precise, let $V^\ddagger$ be the $\mathbb{Q}$-vector space underlying $E^\dagger$. Define a symplectic form

$$\psi^\ddagger(v, w) := \mathrm{Tr}_{F^\dagger/\mathbb{Q}} \left(\sqrt{\lambda}\, \mathrm{Tr}_{E^\dagger/F^\dagger}(v\overline{w})\right)$$

for $v, w \in E^\dagger$. Then $T^\ddagger$ can be identified with the group of $E^\dagger$-linear symplectic similitude of $(V^\ddagger, \psi^\ddagger)$, and $T^\ddagger(\mathbb{Q}_p)$ can be identified with $\mathbb{Q}_p^\times \times \prod_{i=1}^r E^\times_{\mathfrak{p}_i}$. The Hodge map $h^\ddagger$ induces a filtration $0 \subset \mathrm{Fil}^0(V^\ddagger_\mathbb{C}) \subset V^\ddagger_\mathbb{C}$ such that

$$t^\ddagger(e) = \mathrm{tr}\left(e; V^\ddagger_\mathbb{C} / \mathrm{Fil}^0(V^\ddagger_\mathbb{C})\right) = \sum_{\iota \in \Phi} \iota(e)$$

for $e \in E^\dagger$. Since we have identified E (resp. $F^\dagger$) with its embedding through ι_1° (resp. ι_1^1), we can identify $E^\dagger$ with its embedding through $\iota_1^\circ \otimes \iota_1^1$, that is, with $\iota_1^\circ(E) \cdot \iota_1^1(F^\dagger) \subset \mathbb{C}$.

Lemma 5.1.5. *The reflex field $E^{\dagger,\Phi}$ is $E^\dagger$.*

Proof. By definition, $E^{\dagger,\Phi}$ is the field generated by the elements $t^\ddagger(e)$ for all $e \in E^\dagger$. Let $e = (x + y\mu) \times (x' + y'\lambda')$ be an element in $E^\dagger$ with $x, y, x', y' \in F$. Then

$$\begin{aligned} t^\ddagger(e) &= (x + y\mu)(2x') + \sum_{i=2}^d 2\iota_i(x)(\iota_i(x') + \iota_i(y')\lambda') \\ &= 2\,\mathrm{Tr}_{F/\mathbb{Q}}(xx') + 2\,\mathrm{Tr}_{F/\mathbb{Q}}(xy')\lambda' + 2yx'\mu - 2xy'\lambda'. \end{aligned}$$

Therefore, $E^{\dagger,\Phi} = E^{\dagger}$. $\square$

As before, the algebra $E^{\dagger}_p = E^{\dagger} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ decomposes as

$$E^{\dagger}_p = \left(\bigoplus_{i=1}^{r} E^1_i\right) \bigoplus \left(\bigoplus_{i=1}^{r} E^2_i\right) \cong \left(\bigoplus_{i=1}^{r} E_{\mathfrak{p}_i}\right) \bigoplus \left(\bigoplus_{i=1}^{r} E_{\mathfrak{p}_i}\right),$$

and also for its modules. Let π^1_1 be the projection of $E^{\dagger}_p$ to the first factor E^1_1. The additive map $t^{\ddagger}$ extends to a map $t^{\ddagger}_p : E^{\dagger}_p \to E^{\dagger}_p$. From the calculation in the above lemma, we find that for $(e^j_i) = (e^1_1, \ldots, e^1_r; e^2_1, \ldots, e^2_r) \in E^{\dagger}_p$,

$$\pi^1_1 \circ t^{\ddagger}_p((e^j_i)) = \sum_{i=1}^{r} \mathrm{Tr}_{E_{\mathfrak{p}_i}/\mathbb{Q}_p}(e^1_i) + e^1_1 + e^2_1 - \mathrm{Tr}_{E_{\mathfrak{p}_1}/F_{\mathfrak{p}_1}}(e^1_1). \tag{5.6}$$

Let $\mathcal{O}^{\ddagger} = E^{\dagger} \cap \mathcal{O}^{\dagger}$ be the unique maximal $\mathbb{Z}_{(p)}$-order in $E^{\dagger}$ such that $\overline{\mathcal{O}^{\ddagger}} = \mathcal{O}^{\ddagger}$, and $\mathcal{O}^{\ddagger}_{\mathfrak{p}^2_i} = \mathcal{O}_{E_{\mathfrak{p}_i}}$ is the ring of integers, where $\mathcal{O}^{\ddagger}_{\mathfrak{p}^2_i}$ is the projection of $\mathcal{O}^{\ddagger} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ to the E^2_i component. For every abelian variety A over an $E_{\mathfrak{p}^\circ}$-scheme S, which is equipped with an action by $\mathcal{O}^{\ddagger}$, $\mathrm{Lie}_S(A)$ is an $E^{\dagger}_p$-module, and hence decomposes as the direct sum of $\mathrm{Lie}_S(A)^j_i$ ($i = 1, \ldots, r$, $j = 1, 2$). In view of (5.4) and (5.6), we introduce the following trace condition

$$\mathrm{tr}\left(e \in \mathcal{O}_{E_{\mathfrak{p}}} \subset E^2_1; \mathrm{Lie}_S(A)^2_1\right) = e_{\mathfrak{p}^\circ} \in E_{\mathfrak{p}^\circ}; \qquad \mathrm{Lie}_S(A)^2_i = 0, i \geq 2. \tag{5.7}$$

Let

$$K^{\ddagger} = \mathbb{Z}^{\times}_p \times \prod_{i=1}^{r} \mathcal{O}^{\times}_{E_{\mathfrak{p}_i}} \times K^{\ddagger,p}$$

be an open compact subgroup of $T^{\ddagger}(\mathbb{A}_{\mathrm{fin}})$, and we denote $M^{\ddagger}_{00,K^{\ddagger,p}} = M^{\ddagger}_{K^{\ddagger}}$. Let $M^{\ddagger}_{00,K^{\ddagger,p};\mathfrak{p}^\circ}$ be its base change under $\iota_{(p)} \circ (\iota^\circ_1 \otimes \iota^1_1) : E^{\dagger} \to E_{\mathfrak{p}^\circ}$. We fix a sufficiently small compact subgroup $K^{\ddagger,p}$ of $T^{\ddagger}(\mathbb{A}^p_{\mathrm{fin}})$, $M^{\ddagger}_{00,K^{\ddagger,p};\mathfrak{p}^\circ}$ represents the following moduli functor on the category of locally noetherian schemes over $E_{\mathfrak{p}^\circ}$: for every such scheme S, $M^{\ddagger}_{00,K^{\ddagger,p};\mathfrak{p}^\circ}(S)$ is the set of equivalence classes of quadruples $(A, \vartheta, j, \bar{\eta}^p)$, where

- A is an abelian scheme over S of dimension $2d$;
- $\vartheta : A \to A^\vee$ is a prime-to-p polarization;
- $j : \mathcal{O}^{\ddagger} \hookrightarrow \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}$ such that (5.7) is satisfied, and $\vartheta \circ j(e) = j(\bar{e})^\vee \circ \vartheta$ for all $e \in \mathcal{O}^{\ddagger}$;
- $\bar{\eta}^p$ is a $K^{\ddagger,p}$-level structure, that is, a $\pi_1(S, s)$-invariant $K^{\ddagger,p}$-orbit of $E^{\dagger} \otimes \mathbb{A}^p_{\mathrm{fin}}$-linear symplectic similitude $\eta^p : V^{\ddagger} \otimes \mathbb{A}^p_{\mathrm{fin}} \to \mathrm{H}^{\text{ét}}_1(A_s, \mathbb{A}^p_{\mathrm{fin}})$.

The notion of being equivalent is similarly defined as before. Moreover, we can extend this moduli functor to the category of locally noetherian schemes over $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$. We omit the detailed definition. One can similarly prove that the extended moduli functor, denoted by $\mathcal{M}^{\ddagger}_{00,K^{\ddagger,p}}$, is connected, and finite, projective, smooth over $\operatorname{Spec}\mathcal{O}_{E_{\mathfrak{p}^\circ}}$. Therefore, it is isomorphic to $\operatorname{Spec}\mathcal{O}_{E^\natural}$ for some finite unramified extension of local fields $E^\natural/E_{\mathfrak{p}^\circ}$. We fix an embedding $\iota^\natural : E^\natural \hookrightarrow E^0_{\mathfrak{p}^\circ}$. Let $(\mathcal{E},\vartheta,j)$ be the universal object over $\mathcal{M}^{\ddagger}_{00,K^{\ddagger,p}} \times_{\mathcal{O}_{E^\natural},\iota^\natural} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}} \cong \operatorname{Spec}\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$, and denote $\mathcal{Y} = (\mathcal{E}_{p^\infty})^2_1$. Fix a geometric point $s : \mathcal{O}_{E^0_{\mathfrak{p}^\circ}} \hookrightarrow \mathbb{C}$ of characteristic 0 and an $\mathcal{O}^\ddagger$-generator $\mathbf{x}$ of $\mathrm{H}^{\mathrm{Bet}}_1(\mathcal{E}_s,\mathbb{Z}_{(p)})$, where $\mathrm{H}^{\mathrm{Bet}}_\bullet$ is the Betti homology. We call the quadruple $(\mathcal{E},\vartheta,j;\mathbf{x})$ a *basic unitary datum*.

In what follows, we fix a basic unitary datum $(\mathcal{E},\vartheta,j;\mathbf{x})$ once and for all. Since $\operatorname{Spec}\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ is simply connected, $\mathbf{x}$ extends to a unique section $\mathbf{x}^p$ of the lisse $\mathbb{A}^p_{\mathrm{fin}}$-sheaf $\mathrm{H}^{\text{ét}}_1(\mathcal{E},\mathbb{A}^p_{\mathrm{fin}})$ over $\operatorname{Spec}\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$, and determines canonically an element $\mathbf{x}^{\mathfrak{p}}_p$ of $\bigoplus_{i=2}^r (T_p\mathcal{E})^2_i$. For any scheme S over $\operatorname{Spec}\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$, we denote by $\mathbf{x}^p_S$ and $\mathbf{x}^{\mathfrak{p}}_{p,S}$ for the corresponding base change objects, respectively.

Let $(\mathbf{E},\vartheta_{\mathbf{E}},j_{\mathbf{E}})$ be the special fiber of $(\mathcal{E},\vartheta,j)$, where $\mathbf{E}$ is an abelian variety over $\operatorname{Spec}\mathbb{F}$. Let $\mathbf{Y} = (\mathbf{E}_{p^\infty})^2_1$ and $i_{\mathbf{Y}} : \mathcal{O}_{E_{\mathfrak{p}^\circ}} \to \operatorname{End}(\mathbf{Y})$ be the induced $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$-action. Therefore, $\mathbf{Y}$ is the special fiber of $\mathcal{Y}$, which is an $\mathcal{O}_{F_{\mathfrak{p}}}$-module over $\operatorname{Spec}\mathbb{F}$.

5.1.4 The nearby space

In this and the next subsections, we assume that $\mathfrak{p}$ is nonsplit in E, and fix an $\mathbb{F}$-point s in the supersingular locus of the common neutral component of $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ and $\mathcal{M}_{0,K^{\mathfrak{p}}}$, which corresponds to a quintuple $(\mathbf{A},\theta_{\mathbf{A}},i_{\mathbf{A}},\bar{\eta}^p,\bar{\eta}^{\mathfrak{p}}_p)$. Let

$$\breve{V}^\dagger = \operatorname{Mor}\left((\mathbf{E},j_{\mathbf{E}}),(\mathbf{A},i_{\mathbf{A}})\right)\otimes\mathbb{Q},$$ [1]

which is an $E^\dagger$-vector space of dimension 2. The map

$$\operatorname{Mor}\left((\mathbf{E},j_{\mathbf{E}}),(\mathbf{A},i_{\mathbf{A}})\right)\times\operatorname{Mor}\left((\mathbf{E},j_{\mathbf{E}}),(\mathbf{A},i_{\mathbf{A}})\right)\to\mathcal{O}^\ddagger$$

sending $(\breve{x},\breve{y})$ to

$$(\breve{x},\breve{y})' := j_{\mathbf{E}}^{-1}\circ\vartheta_{\mathbf{E}}^{-1}\circ\breve{y}^\vee\circ\theta_{\mathbf{A}}\circ\breve{x}$$

induces a $E^\dagger$-hermitian form on $\breve{V}^\dagger$. If we let $(\mathbf{A}^0,\theta_{\mathbf{A}^0},i_{\mathbf{A}^0})$ be the isogeny class of $(\mathbf{A},\theta_{\mathbf{A}},i_{\mathbf{A}})$, then $\breve{B}^\dagger = \operatorname{End}(\mathbf{A}^0,i_{\mathbf{A}^0})$ is a quaternion algebra over $F^\dagger$. Moreover, the underlying $F^\dagger$-quadratic spaces of $\breve{V}^\dagger$ and $\breve{B}^\dagger$ are isometric. The hermitian form $(-,-)'$ induces a $\mathbb{Q}$-symplectic form on $\breve{V}^\dagger$. If we

[1]Here $(\mathbf{A},i_{\mathbf{A}})$ really means $(\mathbf{A},i_{\mathbf{A}}\mid\mathcal{O}^\ddagger)$, and we apply the similar convention in what follows.

let $\breve{H}^\dagger$ be the corresponding group of symplectic similitude, then $\mathrm{Aut}(\mathbf{A}^0, \theta_{\mathbf{A}^0}, i_{\mathbf{A}^0})$ can be identified with $\breve{H}^\dagger(\mathbb{Q})$. We fix an E-subspace $\breve{V}$ of $\breve{V}^\dagger$ of dimension 2 that is stable under the action of $\breve{H}^\dagger_{\mathrm{der}}(\mathbb{Q})$, and such that the restricted hermitian form $(-,-)'|_{\breve{V}\times\breve{V}}$ takes value in E. Then $\breve{V}^\dagger = \breve{V} \otimes_E E^\dagger$, and $\breve{V}_v$ is isometric to V_v exactly for v away from $\{\iota_1, \mathfrak{p}\}$. Let $\breve{H} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(\breve{V})$. We fix an isometry

$$\gamma^{\mathfrak{p}} = (\gamma^{\mathfrak{p}}_p, \gamma^p) : \breve{V} \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,\mathrm{fin}} \to V \otimes_F \mathbb{A}^{\mathfrak{p}}_{F,\mathrm{fin}}$$

such that

$$\varrho_*(\mathbf{x}^{\mathfrak{p}}_p) \in \bar{\eta}^{\mathfrak{p}}_p(\gamma^{\mathfrak{p}}_p(\varrho)); \qquad \varrho_*(\mathbf{x}^p) \in \bar{\eta}^p(\gamma^p(\varrho))$$

for every element $\varrho \in \breve{V}$. In particular, $\gamma^{\mathfrak{p}}$ identifies $\breve{H}^{\mathfrak{p}}_{\mathrm{fin}}$ with $H^{\mathfrak{p}}_{\mathrm{fin}}$.

5.1.5 Integral special subschemes: minimal level

For every admissible $x \in V \otimes \mathbb{A}_{\mathrm{fin},E}$, we have the subscheme $Z(x)_K$ on M_K that is a special cycle introduced in 3.1.2. For $K = K_{\mathfrak{p},0}K^{\mathfrak{p}}$, let us consider the curve $M_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, which is the base change $M_K \times_E E_{\mathfrak{p}^\circ}$; and its subscheme $Z(x)_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, which is the corresponding base change of $Z(x)_K$.

Consider $x^\dagger \in V^\dagger$ and $h^\dagger \in H^\dagger(\mathbb{A}_{\mathrm{fin}})$, such that

- the $(V^\dagger_p)^2_1$-component of $h^{\dagger,-1}x^\dagger$ is inside Λ^2_1; and
- $K^\dagger h^{\dagger,-1}x^\dagger \cap V \otimes \mathbb{A}_{\mathrm{fin},E}$ (inside $V^\dagger \otimes \mathbb{A}_{\mathrm{fin}}$) contains a K-orbit that has totally positive definite norm in E. Here, $K^\dagger = \mathbb{Z}^\times_p \times \mathcal{O}^\times_{B_{\mathfrak{p}}} \times K^{\dagger,\mathfrak{p}}_p \times K^{\dagger,p}$ and $K = K_{\mathfrak{p},0}K^{\mathfrak{p}}$ for $K^{\mathfrak{p}}$ as in Proposition 5.1.2.

We define a moduli functor $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)^0_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ on the category of locally noetherian schemes over $\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ as follows: for every such scheme S, $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)^0_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}(S)$ is the set of equivalence classes of sextuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p, \varrho_A)$ where

- $(A, \theta, i, \bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p)$ is an element of $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}(S)$;
- $\varrho_A : \mathcal{E} \times_{\mathrm{Spec}\,\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}} S \to A$ is a quasi-homomorphism that satisfies the following conditions:
  1. For any $e \in \mathcal{O}^\ddagger$, the following diagram commutes:

$$\begin{CD}
\mathcal{E} \times_{\mathrm{Spec}\,\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}} S @>{\varrho_A}>> A \\
@V{j(e)}VV @VV{i_A(e)}V \\
\mathcal{E} \times_{\mathrm{Spec}\,\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}} S @>{\varrho_A}>> A;
\end{CD}$$

2. ϱ_A induces a homomorphism from $\mathcal{Y} \times_{\operatorname{Spec} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}} S$ to $(A_{p^\infty})^2_1$;

3. For the geometric point s defining the $K^{\dagger,p}$-level structure, the map $\rho_{A_s,*} : \mathrm{H}^{\text{ét}}_1(\mathcal{E}_s, \mathbb{A}^p_{\text{fin}}) \to \mathrm{H}^{\text{ét}}_1(A_s, \mathbb{A}^p_{\text{fin}})$ sends $\mathbf{x}^p_s$ into $\bar{\eta}^p(h^{\dagger,-1}x^\dagger)$;

4. The map $\rho_{A_s,*} : \bigoplus_{i=2}^r (T_p\mathcal{E}_s)^2_i \to \bigoplus_{i=2}^r (T_pA_s)^2_i \otimes_{\mathcal{O}_{F_{\mathfrak{p}_i}}} F_{\mathfrak{p}_i}$ sends $\mathbf{x}^{\mathfrak{p}}_{p,s}$ into $\bar{\eta}^{\mathfrak{p}}_p(h^{\dagger,-1}x^\dagger)$.

The equivalence relations are defined in the similar way. The evident morphism

$$\mathcal{Z}^\dagger(x^\dagger, h^\dagger)^0_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \to \mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$$

is finite and its image is a 1-dimensional closed subscheme that is stable under the action of the Galois group $\mathrm{Gal}(\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}/\mathcal{O}_{E_{\mathfrak{p}^\circ}})$. Therefore, $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)^0_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ defines a 1-dimensional closed subscheme $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ of $\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$, which depends only on the orbit $K^\dagger h^{\dagger,-1}x^\dagger$. Moreover, by definition, the intersection $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \cap M^\circ_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$ inside $\mathcal{M}^{\dagger,\circ}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ coincides with the special cycle $Z(x)^\circ_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$ if $x \in K^\dagger h^{\dagger,-1}x^\dagger$ has totally positive definite norm in E. Conversely, for every admissible $x \in V \otimes \mathbb{A}_{F,\text{fin}}$ with $x_{\mathfrak{p}} \in \Lambda_{\mathfrak{p}}$, we have a 1-dimensional closed subscheme $\mathcal{Z}(x)_{0,K^{\mathfrak{p}}}$ of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ obtained by the Hecke translation of $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \cap \mathcal{M}^{\dagger,\circ}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$, whose generic fiber is $Z(x)_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. It depends only on the orbit $K_{\mathfrak{p},0}K^{\mathfrak{p}}x$.

From now on, we assume that ($\mathfrak{p}$ is nonsplit in E and) $\epsilon(V_p) = 1$. Following [Car1986, 11], we have the following isomorphisms of sets:

$$[\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]_{\text{ss}}(\mathbb{F}) \cong \breve{H}^\dagger(\mathbb{Q}) \backslash \left(\mathbb{Z} \times \prod_{i=2}^r B^\times_{\mathfrak{p}_i}/K^{\dagger,\mathfrak{p}}_p \times \breve{H}^\dagger(\mathbb{A}^p_{\text{fin}})/K^{\dagger,p} \right);$$
$$[\mathcal{M}_{0,K^{\mathfrak{p}}}]_{\text{ss}}(\mathbb{F}) \cong \breve{H}(\mathbb{Q}) \backslash \breve{H}^{\mathfrak{p}}_{\text{fin}}/K^{\mathfrak{p}},$$

such that in both cases, the neutral double coset corresponds to the point s. Moreover, we have the following lemma.

Lemma 5.1.6. *The special fiber* $[\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]_{\text{sp}}$ *(resp.* $[\mathcal{Z}(x)_{0,K^{\mathfrak{p}}}]_{\text{sp}}$*) of* $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ *(resp.* $\mathcal{Z}(x)_{0,K^{\mathfrak{p}}}$*) is located in the supersingular locus* $[\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]_{\text{ss}}$ *(resp.* $[\mathcal{M}_{0,K^{\mathfrak{p}}}]_{\text{ss}}$*).*

Proof. We only need to prove for $[\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]_{\text{sp}}$. Let $s' = (A, \theta, i, \bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p, \varrho_A)$ be an $\mathbb{F}$-point of $\mathcal{Z}^\dagger(x^\dagger, h^\dagger)_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$. We have a nontrivial homomorphism of $\mathcal{O}_{F_{\mathfrak{p}}}$-modules $\varrho_{A,*} : \mathbf{Y} \to (A_{p^\infty})^2_1 = \mathcal{X}^\dagger_{s'} \cong (\mathcal{X}^{\dagger,\flat}_{s'})^{\oplus 2}$. Therefore, there is at least one projection $(\mathcal{X}^{\dagger,\flat}_{s'})^{\oplus 2} \to \mathcal{X}^{\dagger,\flat}_{s'}$ whose composition with $\varrho_{A,*}$ is nontrivial. Since $\mathbf{Y}$ is formal, $\mathcal{X}^\dagger$ is formal. The lemma follows. $\square$

5.1.6 Remark on the case $F = \mathbb{Q}$

We briefly explain the constructions in the above four subsections in the case $F = \mathbb{Q}$. Therefore, $E/\mathbb{Q}$ is a imaginary quadratic extension. Let $\iota^\circ, \iota^\bullet$ be two different embeddings of E into $\mathbb{C}$ such that $\iota_{(p)} \circ \iota_1^\circ : E \hookrightarrow \mathbb{C}_p$ induces the place $\mathfrak{p}^\circ$. We identify E with a subfield of $\mathbb{C}$ via the embedding ι°. If p is split in E, we denote $\mathfrak{p}^\bullet$ the other place of E above p. We have the hermitian space V over E of dimension 2 and signature $(1,1)$, the unitary group H over $\mathbb{Q}$. For a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_{\mathrm{fin}})$, the Shimura curve $\mathrm{Sh}_K(H,X)$ is a smooth and quasi-projective curve defined over $\iota^\circ(E)$, and is proper if and only if V is anisotropic.

By the Hasse principle, there is a unique up to isometry quaternion algebra B over $\mathbb{Q}$, such that B, as an $\mathbb{Q}$-quadratic space (of dimension 4), is isometric to the underlying F-quadratic space of V, equipped with the quadratic form $\mathrm{Tr}_{E/\mathbb{Q}}(-,-)$, where $(-,-)$ is the hermitian form on V. We identify two quadratic spaces B and V through a fixed isometry and hence V has both left and right multiplication by B. We fix an embedding $E \hookrightarrow B$, through which the action of E on V induced from the left multiplication of B coincides with the E-vector space structure of V. We let $H^\dagger = B^\times$, which is *different* from the case $\mathbb{F} \neq \mathbb{Q}$. Then similarly, we have the Shimura curve $\mathrm{Sh}_{K^\dagger}(H^\dagger, X^\dagger)$ defined over $\mathbb{Q}$. We can view V as a symplectic space over $\mathbb{Q}$ and $H^\dagger$ the group of E-linear symplectic similitude.

Let

$$\mathrm{Sh}(H)_{n,K^p;\mathfrak{p}^\circ} = \mathrm{Sh}_{K_{p,n}K^p}(H,X) \times_E E_{\mathfrak{p}^\circ}; \qquad \mathrm{Sh}(H^\dagger)_{n,K^{\dagger,p};p} = \mathrm{Sh}_{K^\dagger_{p,n}K^{\dagger,p}}(H^\dagger, X^\dagger) \times_{\mathbb{Q}} \mathbb{Q}_p.$$

We have the following proposition that is parallel to Proposition 5.1.2.

Proposition 5.1.7. *Let $K^p \subset H^p_{\mathrm{fin}}$ be an open compact subgroup that is decomposable and sufficiently small. Then there is an open compact subgroup $K^{\dagger,p} \subset H^\dagger(\mathbb{A}^p_{\mathrm{fin}})$, such that the geometric neutral components $\mathrm{Sh}(H)^\circ_{0,K^p;\mathfrak{p}^\circ}$ and $\mathrm{Sh}(H^\dagger)^\circ_{0,K^{\dagger,p};p}$ are defined and isomorphic over $E^0_{\mathfrak{p}^\circ}$.*

We are going to define a moduli functor $\underline{\mathcal{M}^\dagger_{0,K^{\dagger,p}}}$ on the category of locally noetherian schemes over $\mathbb{Z}_p$. There are two case: the anisotropic case, *i.e.* B is division, and the isotropic case, *i.e.* B is isomorphic to the matrix algebra. In the anisotropic case, for every $\mathbb{Z}_p$-scheme S, define $\underline{\mathcal{M}^\dagger_{0,K^{\dagger,p}}}(S)$ to be the set of equivalence classes of quadruples $(A, \theta, i, \bar{\eta}^p)$ where

- A is an abelian surface over S;
- $\theta : A \to A^\vee$ is a prime-to-p polarization;

- $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A)$ of a monomorphism of rings such that $\det(\iota(b); \mathrm{Lie}_S(A)) = \mathrm{Nm}\, b$;

- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure, that is, a $\pi_1(S,s)$-invariant $K^{\dagger,p}$-orbit of $B \otimes \mathbb{A}^p_{\mathrm{fin}}$-linear symplectic similitude $\eta^p : V \otimes \mathbb{A}^p_{\mathrm{fin}} \to \mathrm{H}_1^{\text{ét}}(A_s, \mathbb{A}^p_{\mathrm{fin}})$.

In the isotropic case, for every $\mathbb{Z}_p$-scheme S, define $\underline{\mathcal{M}^\dagger_{0,K^{\dagger,p}}}(S)$ to be the set of equivalence classes of pairs $(A^\flat, \bar{\eta}^{\flat,p})$ where $A^\flat$ is a generalized elliptic curve over S, and $\bar{\eta}^{\flat,p}$ is a $K^{\dagger,p}$-level structure (*cf.* [K-M1985]). In both cases, the equivalence relation is described by prime-to-p isogenies, and $\underline{\mathcal{M}^\dagger_{0,K^{\dagger,p}}}$ is represented by a smooth and projective scheme $\mathcal{M}^\dagger_{0,K^{\dagger,p}}$ over $\mathbb{Z}_p$. Thus we obtain a scheme $\mathcal{M}_{0,K^p}$ that is smooth and projective over $\mathrm{Spec}\, \mathcal{O}_{E_{\mathfrak{p}^\circ}}$ whose generic fiber is (the Baily–Borel compactification of) $\mathrm{Sh}(H)_{0,K^p;\mathfrak{p}^\circ}$. By construction, the neutral components of $\mathcal{M}^\dagger_{0,K^{\dagger,p}} \times_{\mathbb{Z}_p} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ and $\mathcal{M}_{0,K^p} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ are isomorphic. In the isotropic case, we will denote $A = (A^\flat)^\sharp$, $\bar{\eta}^p = (\bar{\eta}^{\flat,p})^\sharp$ when $A^\flat$ is an elliptic curve. Moreover, we have the canonical polarization $\theta : A \to A^\vee$ and the $\mathcal{O}_B = \mathrm{Mat}_2(\mathbb{Z})$-action $i : \mathcal{O}_B \to \mathrm{End}_S(A)$.

When $F = \mathbb{Q}$, in the basic unitary datum $(\mathcal{E}, \vartheta, j; \mathbf{x})$, $\mathcal{E}$ is an elliptic curve over $\mathrm{Spec}\, \mathcal{O}_{E^0_{\mathfrak{p}^\circ}}$ with the principal polarization ϑ and the $\mathcal{O}_E$-action $j : \mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_{E^0_{\mathfrak{p}^\circ}}}(\mathcal{E})$. Let $\mathcal{Y} = (\mathcal{E}_{p^\infty})$. Let $(\mathbf{E}, \vartheta_{\mathbf{E}}, j_{\mathbf{E}})$ be the special fiber of $(\mathcal{E}, \vartheta, j)$, where $\mathbf{E}$ is an elliptic curve over $\mathrm{Spec}\, \mathbb{F}$. Let $\mathbf{Y} = \mathbf{E}_{p^\infty}$ and $i_{\mathbf{Y}} : \mathcal{O}_{E_{\mathfrak{p}^\circ}} \to \mathrm{End}(\mathbf{Y})$ be the induced $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$-action. Therefore, $\mathbf{Y}$ is the special fiber of $\mathcal{Y}$.

We now assume that p is nonsplit in E. For every admissible $x \in V \otimes \mathbb{A}_{\mathrm{fin}}$ with $x_p \in \Lambda_p$, we can similarly define a 1-dimensional closed subscheme $\mathcal{Z}(x)_{0,K^p}$ of $\mathcal{M}_{0,K^{\mathfrak{p}}}$, whose generic fiber is $Z(x)_{0,K^p;\mathfrak{p}^\circ}$. It depends only on the orbit $K_{\mathfrak{p},0}K^{\mathfrak{p}}x$. In the isotropic case, $\mathcal{Z}(x)_{0,K^p}$ is disjoint from the set of cusps. If we further assume that $\epsilon(V_p) = 1$, then we have the following isomorphism of sets:

$$[\mathcal{M}_{0,K^p}]_{\mathrm{ss}}(\mathbb{F}) \cong \check{H}(\mathbb{Q}) \backslash \check{H}^{\mathfrak{p}}_{\mathrm{fin}} / K^p,$$

such that the neutral double coset corresponds to the fix $\mathbb{F}$-point s. Moreover, we have that the special fiber $[\mathcal{Z}(x)_{0,K^p}]_{\mathrm{sp}}$ of $\mathcal{Z}(x)_{0,K^p}$ is located in the supersingular locus $[\mathcal{M}_{0,K^p}]_{\mathrm{ss}}$.

5.2 Local intersection numbers

In this section, we study the formal scheme $\mathscr{N}$ and its special formal subschemes $\mathscr{Z}(\check{x})$. Therefore, $\mathfrak{p}$ will be a finite place of F that is nonsplit in E and such that $\epsilon(V_{\mathfrak{p}}) = 1$.

5.2.1 p-adic uniformization of supersingular locus

Recall that we have fixed an $\mathbb{F}$-point s on the common neutral component of $\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ and $\mathcal{M}_{0,K^{\mathfrak{p}}}$, which corresponds to a quintuple $(\mathbf{A},\theta_{\mathbf{A}},i_{\mathbf{A}},\bar{\eta}^p,\bar{\eta}_p^{\mathfrak{p}})$. Let $\mathbf{X}=(\mathbf{A}_{p^\infty})_1^2$ and $i_{\mathbf{X}}:\mathrm{Mat}_2(\mathcal{O}_{F_\mathfrak{p}})\to\mathrm{End}(\mathbf{X})$ be the induced $\mathrm{Mat}_2(\mathcal{O}_{F_\mathfrak{p}})$-action. Then $\mathbf{X}^\flat$ is a formal $\mathcal{O}_{F_\mathfrak{p}}$-module of dimension 1 and height 2.

We define a moduli functor $\underline{\mathscr{N}}$ on the category of schemes over $\operatorname{Spec}\mathcal{O}_{F_\mathfrak{p}^0}$ where ϖ is locally nilpotent: for every such scheme S, $\underline{\mathscr{N}}(S)$ is the set of equivalence classes of pairs (G,ρ_G) where

- G is an $\mathcal{O}_{F_\mathfrak{p}}$-module over S of dimension 1 and height 2;
- $\rho_G:\mathbf{X}^\flat\times_{\operatorname{Spec}\mathbb{F}}S_{\mathrm{sp}}\to G\times_S S_{\mathrm{sp}}$ is a quasi-isogeny of height 0 (which is in fact an isomorphism). Here, $S_{\mathrm{sp}}=S\times_{\operatorname{Spec}\mathcal{O}_{\widehat{F_\mathfrak{p}^0}}}\operatorname{Spec}\mathbb{F}$.

Two pairs (G,ρ_G) and $(G',\rho_{G'})$ are equivalent if there is an isomorphism $G'\to G$ sending ρ_G to $\rho_{G'}$. Then $\underline{\mathscr{N}}$ is represented by the formal scheme $\mathscr{N}$, which is isomorphic to $\operatorname{Spf}R_{F_\mathfrak{p},2}$, where $R_{F_\mathfrak{p},2}=\mathcal{O}_{\widehat{F_\mathfrak{p}^0}}[[t]]$. Let $\mathscr{N}'=\mathscr{N}\times_{\mathcal{O}_{\widehat{F_\mathfrak{p}^0}}}\mathcal{O}_{\widehat{E_{\mathfrak{p}^\circ}^0}}$.

By the theorem of Serre–Tate, the formal completion of $\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^{\mathfrak{p}}}$) at s is canonically isomorphic to $\mathscr{N}$ (resp. $\mathscr{N}'$). Therefore, if we denote by $[\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]^\wedge_{\mathrm{ss}}$ (resp. $[\mathcal{M}_{0,K^{\mathfrak{p}}}]^\wedge_{\mathrm{ss}}$) the formal completion along the supersingular locus, then we have the following p-adic uniformization:

$$[\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]^\wedge_{\mathrm{ss}}\times_{\mathcal{O}_{F_\mathfrak{p}}}\mathcal{O}_{\widehat{F_\mathfrak{p}^0}}\cong\breve{H}^\dagger(\mathbb{Q})\backslash\left(\mathscr{N}\times\mathbb{Z}\times\prod_{i=2}^r B_{\mathfrak{p}_i}^\times/K_p^{\dagger,\mathfrak{p}}\times\breve{H}^\dagger(\mathbb{A}^p_{\mathrm{fin}})/K^{\dagger,p}\right);$$
$$[\mathcal{M}_{0,K^{\mathfrak{p}}}]^\wedge_{\mathrm{ss}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{\widehat{E_{\mathfrak{p}^\circ}^0}}\cong\breve{H}(\mathbb{Q})\backslash\mathscr{N}'\times\breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}.$$

Such uniformization is a special case of those considered in [RZ1996].

5.2.2 Special formal subschemes

Recall that we have the $E^\dagger$-hermitian space $\breve{V}^\dagger$ and a E-subvector space $\breve{V}$. The obvious morphism

$$\mathrm{Mor}\left((\mathbf{E},j_{\mathbf{E}}),(\mathbf{A},i_{\mathbf{A}})\right)\to\mathrm{Mor}\left((\mathbf{Y},j_{\mathbf{Y}}),(\mathbf{X},i_{\mathbf{X}})\right)$$

in fact identifies the later $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$-module as the maximal lattice Λ^- in $\breve{V}_\mathfrak{p}\cong V^-$. For every

$$\breve{x}\in\mathrm{Mor}\left((\mathbf{Y},j_{\mathbf{Y}}),(\mathbf{X},i_{\mathbf{X}})\right)_{\mathrm{reg}}:=\mathrm{Mor}\left((\mathbf{Y},j_{\mathbf{Y}}),(\mathbf{X},i_{\mathbf{X}})\right)-\{0\},$$

we define a subfunctor $\underline{\mathscr{Z}(\breve{x})}$ of $\underline{\mathscr{N}}$ as follows: for every scheme S in the previously mentioned category, $\underline{\mathscr{Z}(\breve{x})}(S)$ is the set of equivalence classes of $(G,\rho_G)\in\underline{\mathscr{N}}(S)$ such that the following composed homomorphism

$$\left(\mathcal{Y} \times_{\operatorname{Spec} \mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \operatorname{Spec} \mathbb{F}\right) \times_{\operatorname{Spec} \mathbb{F}} S_{\mathrm{sp}} = \mathbf{Y} \times_{\operatorname{Spec} \mathbb{F}} S_{\mathrm{sp}} \xrightarrow{\breve{x}} \mathbf{X} \times_{\operatorname{Spec} \mathbb{F}} S_{\mathrm{sp}} \xrightarrow{\rho^\sharp_G} G^\sharp \times_S S_{\mathrm{sp}}$$

lifts to a homomorphism $\mathcal{Y} \times_{\operatorname{Spec} \mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} S \to G^\sharp$. The functor $\underline{\mathscr{Z}(\breve{x})}$ is represented by a closed formal subscheme $\mathscr{Z}(\breve{x})$ of $\mathscr{N}$. In fact, one can show that it is a relative divisor of $\mathscr{N}$ by the same argument in [KR2011, Proposition 3.5]. We will use the same notation for the base change of $\mathscr{Z}(\breve{x})$ in $\mathscr{N}'$.

Let $\phi = \phi^0_\infty \otimes (\otimes_{v \in \Sigma_{\mathrm{fin}}} \phi_v)$ such that

- $\phi^{\mathfrak{p}}_{\mathrm{fin}} := \otimes_{v \in \Sigma_{\mathrm{fin}} - \{\mathfrak{p}\}} \phi_v$ is in $\mathcal{S}(V(\mathbb{A}^{\mathfrak{p}}_{F,\mathrm{fin}}))^{K^{\mathfrak{p}}}$;
- $\phi_{\mathfrak{p}} = \phi^0_{\mathfrak{p}}$ is the characteristic function of the selfdual lattice Λ^+ of $V_{\mathfrak{p}}$;
- $\phi(0) = 0$.

Let $g = (g_v) \in H'(\mathbb{A}_F)$ such that $g_{\mathfrak{p}} \in n(b_{\mathfrak{p}})\mathcal{K}'_{\mathfrak{p}}$ for some unipotent element $n(b_{\mathfrak{p}}) \in N'(F_{\mathfrak{p}})$. Consider the generating series

$$\begin{aligned} Z_\phi(g) &= \sum_{x \in K \backslash V(\mathbb{A}_{F,\mathrm{fin}}) - \{0\}} (\omega_\chi(g)\phi)(x) Z(x)_K \\ &= \sum_{x \in K \backslash V(\mathbb{A}_{F,\mathrm{fin}}) - \{0\}} \psi_{\mathfrak{p}}(b_{\mathfrak{p}} T(x)) \left(\phi^0_{\mathfrak{p}} \otimes (\omega_\chi(g^{\mathfrak{p}})\phi^{\mathfrak{p}})\right)(x) Z(x)_{K_{\mathfrak{p},0}K^{\mathfrak{p}}}. \end{aligned}$$

Define

$$\mathcal{Z}_\phi(g)_{0,K^{\mathfrak{p}}} = \sum_{x \in K \backslash V(\mathbb{A}_{F,\mathrm{fin}}) - \{0\}} \psi_{\mathfrak{p}}(b_{\mathfrak{p}} T(x)) \left(\phi^0_{\mathfrak{p}} \otimes (\omega_\chi(g^{\mathfrak{p}})\phi^{\mathfrak{p}})\right)(x) \mathcal{Z}(x)_{0,K^{\mathfrak{p}}},$$

whose generic fiber is the base change of $Z_\phi(g)$ to $E_{\mathfrak{p}^\circ}$. Moreover, its completion at the $\mathbb{F}$-point s is

$$[\mathcal{Z}_\phi(g)_{0,K^{\mathfrak{p}}}]^\wedge_s = \sum_{\breve{x} \in \breve{V}} \psi_{\mathfrak{p}}(b_{\mathfrak{p}} T(\breve{x})) \left(\breve{\phi}^0_{\mathfrak{p}} \otimes (\omega_\chi(g^{\mathfrak{p}})\phi^{\mathfrak{p}})\right)(\breve{x}) \mathscr{Z}(\breve{x}). \tag{5.8}$$

Here, $\breve{\phi}^0_{\mathfrak{p}}$ is the characteristic function of $\operatorname{Mor}((\mathbf{Y}, j_{\mathbf{Y}}), (\mathbf{X}, i_{\mathbf{X}}))$, and in particular, the notation $\mathscr{Z}(\breve{x})$ makes sense since $\phi(0) = 0$.

5.2.3 A formula for local intersection multiplicity

In this and the next subsections, we will assume that $\mathfrak{p}$ is not divided by 2 and is *inert* in E. For every pair $(\breve{x}_1, \breve{x}_2) \in \operatorname{Mor}((\mathbf{Y}, j_{\mathbf{Y}}), (\mathbf{X}, i_{\mathbf{X}}))^2_{\mathrm{reg}}$ that are linearly independent, the formal divisors $\mathscr{Z}(\breve{x}_1)$ and

$\mathscr{Z}(\breve{x}_2)$ intersects properly at the unique closed point of $\mathscr{N}$. We would like to calculate the intersection number $\mathscr{Z}(\breve{x}_1)\cdot\mathscr{Z}(\breve{x}_2)$. Assume that $(\breve{y}_1,\breve{y}_2)=(\breve{x}_1,\breve{x}_2)g$ for some $g\in \mathrm{GL}_2(\mathcal{O}_{E_{\mathfrak{p}^\circ}})$ such that $(\breve{y}_1,\breve{y}_2)$ has the moment matrix

$$T\left((\breve{y}_1,\breve{y}_2)\right)=\begin{pmatrix}\varpi^a & \\ & \varpi^b\end{pmatrix}$$

with a nonnegative even and b positive odd. If we denote by $\mathrm{Def}(\mathbf{X}^\flat,(\breve{x}_1,\breve{x}_2))$ the subring of $\mathrm{Def}(\mathbf{X}^\flat)=\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}[[t]]$ where $(\breve{x}_1,\breve{x}_2)$ deforms, then

$$\mathscr{Z}(\breve{x}_1)\cdot\mathscr{Z}(\breve{x}_2)=\mathrm{length}_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}}\mathrm{Def}(\mathbf{X}^\flat,(\breve{x}_1,\breve{x}_2))=\mathrm{length}_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}}\mathrm{Def}(\mathbf{X}^\flat,(\breve{y}_1,\breve{y}_2))=\mathscr{Z}(\breve{y}_1)\cdot\mathscr{Z}(\breve{y}_2).$$

Therefore, we only need to study the intersection number $\mathscr{Z}(\breve{y}_1)\cdot\mathscr{Z}(\breve{y}_2)$.

Let $\mathbf{Y}^\tau$ be the unique (up to isomorphism) formal $\mathcal{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2 with an $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$ action $j_{\mathbf{Y}^\tau}$, such that

- As $\mathcal{O}_{F_{\mathfrak{p}}}$-modules, there is an isomorphism $\mathbf{Y}\cong\mathbf{Y}^\tau$ (and we fix such an isomorphism); and
- $j_{\mathbf{Y}^\tau}$ is given by the composition $\mathcal{O}_{E_{\mathfrak{p}^\circ}}\xrightarrow{\tau}\mathcal{O}_{E_{\mathfrak{p}^\circ}}\xrightarrow{j_{\mathbf{Y}}}\mathrm{End}(\mathbf{Y})\cong\mathrm{End}(\mathbf{Y}^\tau)$.

By [KR2011, Lemma 4.2], there is an isomorphism

$$\rho_{\mathbf{X}}:\mathbf{Y}\times\mathbf{Y}^\tau\to\mathbf{X}$$

that commutes with $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$-actions, such that as elements of $\mathrm{Hom}_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}(\mathbf{Y},\mathbf{Y}\times\mathbf{Y}^\tau)$,

$$\rho_{\mathbf{X}}^{-1}\circ\breve{y}_\alpha=\begin{cases}\mathrm{inc}_\alpha\circ\Pi^a & \alpha=1;\\ \mathrm{inc}_\alpha\circ\Pi^b & \alpha=2,\end{cases}$$

where

- inc_α $(\alpha=1,2)$ denotes the inclusion of $\mathbf{Y}$ into the α-th component of $\mathbf{Y}\times\mathbf{Y}^\tau\cong\mathbf{Y}\times\mathbf{Y}$; and
- Π is a fixed uniformizer of the division algebra $\mathrm{End}(\mathbf{Y})$.

In what follows, we will identify $\mathbf{X}$ with $\mathbf{Y}\times\mathbf{Y}^\tau$ via the isomorphism $\rho_{\mathbf{X}}$.

For an integer $l\geq 0$, let F_l be a quasi-canonical lifting of level l, which is an $\mathcal{O}_{F_{\mathfrak{p}}}$-module over $\mathrm{Spf}\,\mathcal{O}_{\widehat{F^l_{\mathfrak{p}}}}$, unique up to the Galois action (*cf.* [Gro1986]). Therefore, it defines a morphism $\mathrm{Spf}\,\mathcal{O}_{\widehat{F^l_{\mathfrak{p}}}}\to\mathscr{N}$ that is a closed immersion. Let $\mathscr{Z}_l$ be the divisor of $\mathscr{N}$ defined by the image, which is independent of F_l we choose. We have the following proposition generalizing [KR2011, Proposition 8.1] from $\mathbb{Q}_p$ to $F_{\mathfrak{p}}$.

Proposition 5.2.1. *As divisors on $\mathcal{N}$, we have*

$$\mathscr{Z}(\breve{y}_1) = \sum_{\substack{l=0 \\ \text{even}}}^{a} \mathscr{Z}_l; \qquad \mathscr{Z}(\breve{y}_2) = \sum_{\substack{l=1 \\ \text{odd}}}^{b} \mathscr{Z}_l.$$

Proof. The original proof of [KR2011, Proposition 8.1] works again for one direction. Namely,

$$\sum_{\substack{l=0 \\ \text{even}}}^{a} \mathscr{Z}_l \leq \mathscr{Z}(\breve{y}_1); \qquad \sum_{\substack{l=1 \\ \text{odd}}}^{b} \mathscr{Z}_l \leq \mathscr{Z}(\breve{y}_2).$$

To prove the other direction, we need only to prove that the intersection multiplicities of both sides in two cases with the special fiber $\mathcal{N}_{\text{sp}} = \operatorname{Spf} \mathbb{F}[[t]]$ are the same. For the left-hand side, we have

$$\sum_{\substack{l=0 \\ \text{even}}}^{a} \mathscr{Z}_l \cdot \mathcal{N}_{\text{sp}} = \sum_{\substack{l=0 \\ \text{even}}}^{a} [\mathcal{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{l}] = \frac{q^{a+1}-1}{q-1};$$
$$\sum_{\substack{l=1 \\ \text{odd}}}^{b} \mathscr{Z}_l \cdot \mathcal{N}_{\text{sp}} = \sum_{\substack{l=1 \\ \text{odd}}}^{b} [\mathcal{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{l}] = \frac{q^{b+1}-1}{q-1},$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$. Then the assertion follows from the following proposition that generalizes [KR2011, Proposition 8.2]. $\square$

Proposition 5.2.2. *For $\breve{y} \in \operatorname{Hom}_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}(\mathbf{Y}, \mathbf{Y} \times \mathbf{Y}^{\tau})$, the intersection multiplicity*

$$\mathscr{Z}(\breve{y}) \cdot \mathcal{N}_{\text{sp}} = \frac{q^{v+1}-1}{q-1},$$

where $v \geq 0$ is the valuation of $(\breve{y}, \breve{y})'$, i.e. $(\breve{y}, \breve{y})' \in \varpi^v \mathcal{O}_{F_{\mathfrak{p}}}^{\times}$.

We keep the assumptions and notations in the above subsection. The results in [ARG2007] cited in the proof of [KR2011, Proposition 8.4] also work for $F_{\mathfrak{p}}$, not just $\mathbb{Q}_p$. Therefore, for $0 < l \leq b$ odd, we have

$$\mathscr{Z}(\breve{y}_1) \cdot \mathscr{Z}_l = \begin{cases} \frac{q^{a+1}-1}{q-1} & a < l; \\ \frac{q^l-1}{q-1} + \frac{1}{2}(a+1-l)[\mathcal{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{l}] & a \geq l. \end{cases}$$

Summing over l, we get the following *local arithmetic Siegel-Weil formula* at a good finite place.

Theorem 5.2.3. *For every pair* $(\breve{x}_1, \breve{x}_2) \in \mathrm{Mor}\,((\mathbf{Y}, j_{\mathbf{Y}}), (\mathbf{X}, i_{\mathbf{X}}))^2_{\mathrm{reg}}$ *that are linearly independent, the intersection multiplicity* $\mathscr{Z}(\breve{x}_1) \cdot \mathscr{Z}(\breve{x}_2)$ *depends only on the* $\mathrm{GL}_2(\mathcal{O}_{E_{\mathfrak{p}^\circ}})$*-equivalence class of the moment matrix* $T = T((\breve{x}_1, \breve{x}_2))$*. Moreover, if*

$$T \sim \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} \qquad 0 \leq a < b,$$

then we have

$$H_{\mathfrak{p}}(T) := \mathscr{Z}(\breve{x}_1) \cdot \mathscr{Z}(\breve{x}_2) = \frac{1}{2} \sum_{l=0}^{a} q^l (a + b + 1 - 2l),$$

where q *is the cardinality of the residue field of* $F_{\mathfrak{p}}$*.*

5.2.4 Proof of Proposition 5.2.2

We generalize the proof of [KR2011, Proposition 8.2] to the case $F_{\mathfrak{p}} \neq \mathbb{Q}_p$, again by exploring the theory of windows and displays of p-divisible groups developed by T. Zink in [Zin2001, Zin2002]. In the proof, we simply write $F = F_{\mathfrak{p}}$, $E = E_{\mathfrak{p}^\circ}$. Moreover, we let e and f be the ramification index and the extension degree of residue fields of $F/\mathbb{Q}_p$, respectively. In particular, $q = p^f$. Let $R = \mathbb{F}[[t]]$ and $A = W[[t]]$, where $W = W(\mathbb{F})$ is the Witt ring. We extend the Frobenius automorphism σ on W to A by letting $\sigma(t) = t^p$. For every $l \geq 1$, we let $R_l = R/t^l$ and $A_l = A/t^l$. Then A (resp. A_l) is a frame of R (resp. R_l). The category of formal p-divisible groups over R is equivalent to the category of pairs (M, α) where

- M is a free A-module of finite rank; and
- $\alpha : M \to M^{(\sigma)} := A^\sigma \otimes_A M$ is an A-linear injective homomorphism such that $\operatorname{coker} \alpha$ is a free R-module.

The baby case would be the one where $f = 1$. Consider the p-divisible group $\mathbf{Y}$ over $\mathbb{F}$ of dimension 1 and (absolute) height $2e$ with action by $\mathcal{O}_E$. It corresponds to the pair (N, β), where

$$N = N_0 \oplus N_1 = \mathcal{O}_{\widehat{F^0}} n_0 \oplus \mathcal{O}_{\widehat{F^0}} n_1$$

is the $\mathbb{Z}/2$-graded free $\mathcal{O}_{\widehat{F^0}} = \mathcal{O}_F \otimes_{\mathbb{Z}_p} W$-module of rank 2 (that is a free W-module of rank $2e$), and $\beta(n_0) = \varpi n_1$, $\beta(n_1) = n_0$. We extend the Frobenius automorphism on W to $\mathcal{O}_{\widehat{F^0}}$ $\mathcal{O}_F$-linearly. Similarly as in the proof of [KR2011, Proposition 8.2], the p-divisible group $\mathbf{X} = \mathbf{Y} \times \mathbf{Y}^\tau$ over $\mathbb{F}$ corresponds to (M, α) described there, and its universal deformation is (M, α_t). The only difference is that we should replace p by ϖ. The rest of the proof follows in the same way.

Now we treat the general case and hence assume that $f \geq 2$. Consider the p-divisible group $\mathbf{Y}$ over $\mathbb{F}$. It corresponds to the pair (N, β), where N is a $\mathbb{Z}/2$-graded free $\mathcal{O}_F \otimes_{\mathbb{Z}_p} W$-module of rank 2. Since

$$\mathcal{O}_F \otimes_{\mathbb{Z}_p} W = \bigoplus_{j=0}^{f-1} \mathcal{O}_{\widehat{F^0}}^{(\sigma^j)} := \bigoplus_{j=0}^{f-1} \mathcal{O}_F \otimes_{W(k),\sigma^j} W,$$

where k is the residue field of F, we can write

$$N = \left(\bigoplus_{j=0}^{f-1} \mathcal{O}_{\widehat{F^0}}^{(\sigma^j)} e_{0,j} \right) \bigoplus \left(\bigoplus_{j=0}^{f-1} \mathcal{O}_{\widehat{F^0}}^{(\sigma^j)} e_{1,j} \right),$$

and

- $\beta(e_{i,j}) = e_{i,j+1}$ for $i = 1, 2$, $0 \leqslant j < f - 1$;
- $\beta(e_{0,f-1}) = e_{1,0}$;
- $\beta(e_{1,f-1}) = \varpi e_{0,0}$.

Similarly, the p-divisible group $\mathbf{Y}^\tau$ corresponds to (N^τ, β^τ), which we write as

$$N^\tau = \left(\bigoplus_{j=0}^{f-1} \mathcal{O}_{\widehat{F^0}}^{(\sigma^j)} e^\tau_{0,j} \right) \bigoplus \left(\bigoplus_{j=0}^{f-1} \mathcal{O}_{\widehat{F^0}}^{(\sigma^j)} e^\tau_{1,j} \right),$$

and

- $\beta^\tau(e^\tau_{i,j}) = e^\tau_{i,j+1}$ for $i = 0, 1$, $0 \leqslant j < f - 1$;
- $\beta^\tau(e^\tau_{1,f-1}) = e^\tau_{0,0}$;
- $\beta^\tau(e^\tau_{0,f-1}) = \varpi e^\tau_{1,0}$.

We extend (N, β) (resp. (N^τ, β^τ)) to $\mathbb{F}[[t]]$ by scalars, which we still denote by the same notations.

The p-divisible group $\mathbf{X}$ corresponds to the direct sum $(M, \alpha) := (N, \beta) \oplus (N^\tau, \beta^\tau)$. Under the basis

$$\{e_{0,0}, e^\tau_{1,0}, \ldots, e_{0,f-1}, e^\tau_{1,f-1}; e_{1,0}, e^\tau_{0,0}, \ldots, e_{1,f-1}, e^\tau_{0,f-1}\},$$

the matrix of α is

$$\alpha = \begin{pmatrix} & & & & & & \varpi & \\ & & & & & & & \varpi \\ 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \end{pmatrix}.$$

Let (M, α_t) correspond to the universal deformation of $(\mathbf{X}, i_{\mathbf{X}})$ over $\mathbb{F}[[t]]$. Then under the same basis,

$$\alpha_t = \begin{pmatrix} 1 & & & & & t & & \\ & 1 & & & -t & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \cdot \alpha.$$

Explicitly, we have

$$\begin{aligned} \alpha_t(e_{i,j}) &= e_{i,j+1} \qquad i = 0, 1 \text{ and } j = 0, \ldots, f-2; \\ \alpha_t(e_{0,f-1}) &= e_{1,0} - t e^{\tau}_{1,0}; \\ \alpha_t(e_{1,f-1}) &= \varpi e_{0,0}; \\ \alpha_t(e^{\tau}_{i,j}) &= e^{\tau}_{i,j+1} \qquad i = 0, 1 \text{ and } j = 0, \ldots, f-2; \\ \alpha_t(e^{\tau}_{1,f-1}) &= e^{\tau}_{0,0} + t e_{0,0}; \\ \alpha_t(e^{\tau}_{0,f-1}) &= \varpi e^{\tau}_{1,0}. \end{aligned}$$

If we denote by $\sigma^k(\alpha): M^{(\sigma^k)} \to M^{(\sigma^{k+1})}$ the induced homomorphism for $k \geqslant 0$. Then formally, we have

$$
\begin{aligned}
\sigma^k(\alpha)^{-1}(e_{i,j}) &= e_{i,j-1} \qquad i = 1,2 \text{ and } j = 1,\dots,f-1;\\
\sigma^k(\alpha)^{-1}(e_{0,0}) &= \frac{1}{\varpi} e_{1,f-1};\\
\sigma^k(\alpha)^{-1}(e_{1,0}) &= e_{0,f-1} + \frac{t^{p^k}}{\varpi} e^{\tau}_{0,f-1};\\
\sigma^k(\alpha)^{-1}(e^{\tau}_{i,j}) &= e^{\tau}_{i,j-1} \qquad i = 1,2 \text{ and } j = 1,\dots,f-1;\\
\sigma^k(\alpha)^{-1}(e^{\tau}_{1,0}) &= \frac{1}{\varpi} e^{\tau}_{0,f-1};\\
\sigma^k(\alpha)^{-1}(e^{\tau}_{0,0}) &= e^{\tau}_{1,f-1} - \frac{t^{p^k}}{\varpi} e_{1,f-1}.
\end{aligned}
$$

Now let $\breve{y}$ correspond to the graded A_1-linear homomorphism $\gamma: N \otimes_A A_1 \to M$. Then the length $\mathscr{X}(\breve{y}) \cdot \mathscr{N}_{\mathrm{sp}}$ of the deformation space of γ is the maximal number a such that there exists a diagram

$$
\begin{tikzcd}
N \arrow[r, "\beta"] \arrow[d, "\tilde{\gamma}"'] & N^{(\sigma)} \arrow[d, "\tilde{\gamma}^{(\sigma)}"] \\
M \arrow[r, "\alpha_t"] & M^{(\sigma)},
\end{tikzcd}
$$

which commutes modulo t^a, and $\tilde{\gamma}$ lifts γ.

Case i: $v = 2r$ **is even.** We may assume that $\gamma = \varpi^r \mathrm{inc}_1$ that is represented by the following $4f \times 2f$ matrix

$$
X(0) = \begin{pmatrix}
\varpi^r & 0 & \cdots & 0\\
0 & 0 & \cdots & 0\\
0 & \varpi^r & \cdots & 0\\
0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \varpi^r\\
0 & 0 & \cdots & 0
\end{pmatrix}.
$$

If $r = 0$, in order to lift γ to $\tilde{\gamma} \mod t^p$, we search for a $4f \times 2f$ matrix $X(1)$ with entries in A_p such that $X(1) \equiv X(0)$ in A_1 and satisfies

$$\alpha_t \circ X(1) = \sigma(X(1)) \circ \beta.$$

But $\sigma(X(1)) = \sigma(X(0)) = X(0)$. Therefore, we need to find the largest integer $a \leq p$ such that $\alpha_t^{-1} \circ X(0) \circ \beta$ has integral entries $\mod t^a$. Since the entry at the place $(e_{0,f-1}, e^\tau_{0,f-1})$ is $\frac{t}{\varpi}$, the largest a is simply 1. It follows that when $v = r = 0$, the proposition holds.

If $r > 0$, we first show that we can lift γ to $\tilde{\gamma} \mod t^{q^{2r}}$. By induction, we introduce $X(k)$ for $k \geq 1$, *i.e.* the one satisfying $X(k+1) \equiv X(k)$ in A_{p^k}, and

$$\alpha_t \circ X(k+1) = \sigma(X(k+1)) \circ \beta.$$

Since $\sigma(X(k+1)) = \sigma(X(k))$, we formally have

$$X(k+1) = \alpha_t^{-1} \circ \sigma(X(k)) \circ \beta.$$

We need to show that

$$X(2rf) = \alpha_t^{-1} \circ \sigma(\alpha_t)^{-1} \circ \cdots \circ \sigma^{2rf-1}(\alpha_t)^{-1} \circ X(0) \circ \beta^{2rf}$$

has integral entries. Let $x_{i,j;i',j'}$ (resp. $x^\tau_{i,j;i',j'}$) be the entry of $X(2rf) \mod \varpi$ at the place $(e_{i,j}, e_{i',j'})$ (resp. $(e_{i,j}, e^\tau_{i',j'})$). Then among all these terms, the only nonzero terms are

$$\begin{aligned} x^\tau_{0,j;0,j} &= (-1)^{r-1} t^{p^{f-1-j}(q^{2r-2}+q^{2r-3}+\cdots+1)} \qquad j = 0, \ldots, f-1; \\ x_{1,j;1,j} &= (-1)^r t^{p^{f-1-j}(q^{2r-1}+q^{2r-2}+\cdots+1)} \qquad j = 0, \ldots, f-1, \end{aligned}$$

which implies that we can lift γ to $\tilde{\gamma} \mod t^{q^{2r}}$. Next, we consider the lift of γ to $\tilde{\gamma} \mod t^{pq^{2r}}$. Therefore, we consider the matrix

$$X(2rf+1) = \alpha_t^{-1} \circ \sigma(X(2rf)) \circ \beta.$$

It has exactly one entry that is not integral: the place $(e_{0,f-1}, e^\tau_{0,f-1})$, whose non-integral part is

$$\frac{t}{\varpi}(-1)^r t^{p \cdot p^{f-1}(q^{2r-1}+q^{2r-2}+\cdots+1)} = \frac{(-1)^r}{\varpi} t^{q^{2r}+q^{2r-1}+\cdots+1}.$$

It turns out that the length $\mathscr{L}(\check{y}) \cdot \mathscr{N}_{\mathrm{sp}}$ is exactly $\frac{q^{2r+1}-1}{q-1} = \frac{q^{v+1}-1}{q-1}$.

Case ii: $v = 2r+1$ is odd. We may assume that $\gamma = \varpi^r \mathrm{inc}_2 \circ \Pi$, where Π is the endomorphism of $\mathbf{Y}$ determined by $\Pi(e_{0,j}) = e_{0,j}$ and $\Pi(e_{1,j}) = \varpi e_{0,j}$ for $j = 0, \ldots, f-1$. Then γ is represented by the following $4f \times 2f$ matrix

$$\begin{pmatrix}
 & & & 0 & \cdots & 0 \\
 & & & \varpi^{r+1} & \cdots & 0 \\
 & & & \vdots & \ddots & \vdots \\
 & & & 0 & \cdots & 0 \\
 & & & 0 & \cdots & \varpi^{r+1} \\
0 & \cdots & 0 & & & \\
\varpi^{r} & \cdots & 0 & & & \\
\vdots & \ddots & \vdots & & & \\
0 & \cdots & 0 & & & \\
0 & \cdots & \varpi^{r} & & &
\end{pmatrix}.$$

Similarly, we introduce the matrix $Y(k)$ for $k \geq 0$. We first show that γ can be lifted to $\tilde{\gamma}$ mod $t^{q^{2r+1}}$, *i.e.* the matrix

$$Y((2r+1)f) = \alpha_t^{-1} \circ \sigma(\alpha_t)^{-1} \circ \cdots \circ \sigma^{(2r+1)f-1}(\alpha_t)^{-1} \circ Y(0) \circ \beta^{(2r+1)f}$$

has integral entries. Let $y_{i,j;i',j'}$ (resp. $y^{\tau}_{i,j;i',j'}$) be the entry of $Y((2r+1)f) \mod \varpi$ at the place $(e_{i,j}, e_{i',j'})$ (resp. $(e_{i,j}, e^{\tau}_{i',j'})$). Then among all these terms, the only nonzero terms are

$$\begin{aligned}
y^{\tau}_{0,j;0,j} &= (-1)^r t^{p^{f-1-j}(q^{2r-1}+q^{2r-2}+\cdots+1)} \qquad & j = 0, \ldots, f-1; \\
y_{1,j;1,j} &= (-1)^{r+1} t^{p^{f-1-j}(q^{2r}+q^{2r-1}+\cdots+1)} \qquad & j = 0, \ldots, f-1,
\end{aligned}$$

which implies that we can lift γ to $\tilde{\gamma} \mod t^{q^{2r+1}}$. Next we consider the lift of γ to $\tilde{\gamma} \mod t^{pq^{2r+1}}$. Therefore, we consider the matrix

$$Y((2r+1)f+1) = \alpha_t^{-1} \circ \sigma(Y((2r+1)f)) \circ \beta.$$

It has exactly one entry which is not integral: the place $(e_{0,f-1}, e^{\tau}_{0,f-1})$, whose non-integral part is

$$\frac{t}{\varpi}(-1)^{r+1} t^{p \cdot p^{f-1}(q^{2r}+q^{2r-1}+\cdots+1)} = \frac{(-1)^{r+1}}{\varpi} t^{q^{2r+1}+q^{2r}+\cdots+1}.$$

It turns out that the length $\mathscr{X}(\breve{y}) \cdot \mathscr{N}_{\mathrm{sp}}$ is exactly $\frac{q^{2r+2}-1}{q-1} = \frac{q^{v+1}-1}{q-1}$. Therefore, the proposition is proved.

5.3 Comparison

5.3.1 Non-archimedean Whittaker integrals

We calculate certain Whittaker integrals $W_T(s,g,\Phi)$ and their derivatives (at $s=0$) at a non-archimedean place when T is of rank 2.

Assume that E/F is unramified and $p>2$. We fix a selfdual $\mathcal{O}_E$-lattice Λ^+ in V^+ and let $\phi^{0+} \in \mathcal{S}(V^+)$ (resp. $\Phi^{0+} \in \mathcal{S}((V^+)^2)$) be the characteristic function of Λ^+ (resp. $(\Lambda^+)^2$). Let ψ be the unramified character of F. For $T \in \mathrm{Her}_2(E)$ that is nonsingular, we consider the Whittaker integral

$$W_T(s,g,\Phi^{0+}) = \int_{\mathrm{Her}_2(E)} \varphi_{\Phi^{0+},s}(wn(u)g)\psi_T(n(u))^{-1}\mathrm{d}u \tag{5.9}$$

for $\mathrm{Re}\, s>1$, where $\mathrm{d}u$ is the selfdual measure with respect to ψ. Write $g=n(b)m(a)k$ under the Iwasawa decomposition of H''. Then

$$\begin{aligned}
(5.9) &= \int_{\mathrm{Her}_2(E)} \left(\omega''_{\mathbf{1}}(wn(u)n(b)m(a)k)\Phi^{0+}\right)(0)\lambda_P(wn(u)n(b)m(a)k)^s\psi(-\mathrm{tr}\,Tu)\mathrm{d}u \\
&= \psi(\mathrm{tr}\,Tb)\int_{\mathrm{Her}_2(E)} \lambda_P(wn(u)m(a))^s\psi(-\mathrm{tr}\,Tu)\mathrm{d}u \\
&= \psi(\mathrm{tr}\,Tb)|\det a|_E^{1-s}W_{{}^t a^\tau T a}(s,e,\Phi^{0+}).
\end{aligned}$$

Therefore, we only need to consider the integral $W_T(s,e,\Phi^{0+})$. If T is not in $\mathrm{Her}_2(\mathcal{O}_E)$, then $W_T(s,e,\Phi^{0+})$ is identically 0. For $T\in\mathrm{Her}_2(\mathcal{O}_E)$, it is well-known (e.g., [Kud1997, Appendix]) that for an integer $r>1$, $W_T(r,e,\Phi^{0+}) = \gamma_{V^+}\alpha_F(\mathbf{1}_{2+r},T)$, where γ_{V^+} is the Weil constant and α_F is the classical representation density (for hermitian matrices). By [Hir1999], we see that for $r\geq 0$,

$$\alpha_F(\mathbf{1}_{2+r},T) = P_F(\mathbf{1}_2,T;(-q)^{-r})$$

for a polynomial $P_F(\mathbf{1}_2,T;X)\in\mathbb{Q}[X]$. By analytic continuation, we see that

$$W_T(s,e,\Phi^{0+}) = \gamma_{V^+}P_F(\mathbf{1}_2,T;(-q)^{-s}).$$

If $\mathrm{ord}(\det T)$ is odd, *i.e.* T can not be represented by V^{+}, then $W_T(0, e, \Phi^{0+}) = P_F(\mathbf{1}_2, T; 1) = 0$. Taking derivative at $s = 0$, we have

$$W'_T(0, e, \Phi^{0+}) = -\gamma_{V^+} \log q \cdot \frac{\mathrm{d}}{\mathrm{d}X} P_F(\mathbf{1}_2, T; X) \mid_{X=1} .$$

Moreover, we have the following results.

Proposition 5.3.1 (Hironaka, [Hir1999])**.** *Suppose that T is $\mathrm{GL}_2(\mathcal{O}_E)$-equivalent to $\mathrm{diag}[\varpi^a, \varpi^b]$ with $0 \leq a < b$. Then*

$$P_F(\mathbf{1}_2, T; X) = (1 + q^{-1}X)(1 - q^{-2}X) \sum_{l=0}^{a} (qX)^l \left(\sum_{k=0}^{a+b-2l} (-X)^k \right).$$

Corollary 5.3.2. *If $a + b$ is odd, then*

$$W'_T(0, e, \Phi^{0+}) = \gamma_{V^+} b_2(0)^{-1} \log q \cdot \frac{1}{2} \sum_{l=0}^{a} q^l (a + b - 2l + 1).$$

5.3.2 Comparison on Shimura curves

In this subsection, we calculate the local height pairing $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ}$ at a finite place $\mathfrak{p}^\circ$ of E that is *good*. Recall that we have a (compactified) Shimura curve M_K constructed from the hermitian space $\mathbb{V}$, and assume that K is sufficiently small and decomposable. We also assume that ϕ_α ($\alpha = 1, 2$) are decomposable and $K = K_{\mathfrak{p}} K^{\mathfrak{p}}$-invariant.

Let $\mathsf{S} \subset \Sigma_{\mathrm{fin}}$ be a finite subset of cardinality at least 2, such that for every place $\mathfrak{p} \in \Sigma_{\mathrm{fin}} - \mathsf{S}$, we have

- $\mathfrak{p} \nmid 2$, $\mathfrak{p}$ is inert or split in E;
- $\epsilon(\mathbb{V}_{\mathfrak{p}}) = 1$;
- $\phi_{\alpha,\mathfrak{p}} = \phi^0_{\mathfrak{p}}$ ($\alpha = 1, 2$) are the characteristic function of a selfdual lattice $\Lambda_{\mathfrak{p}} = \Lambda^+ \subset \mathbb{V}_{\mathfrak{p}}$;
- $K_{\mathfrak{p}} = K_{\mathfrak{p},0}$ is the subgroup of $\mathrm{U}(\mathbb{V}_{\mathfrak{p}})$ stabilizing $\Lambda_{\mathfrak{p}}$, *i.e.* $K_{\mathfrak{p}}$ is a hyperspecial maximal compact subgroup;
- χ and ψ are both unramified at $\mathfrak{p}$.

We say a finite place $\mathfrak{p}^\circ$ of E is *good* if it is not lying over some place in S. Assume that $\phi_\alpha(0) = 0$. Consider the generating series $Z_{\phi_\alpha}(g_\alpha)$ for $\alpha = 1, 2$. Write $g_{\alpha,\mathfrak{p}} = n(b_{\alpha,\mathfrak{p}}) m(a_{\alpha,\mathfrak{p}}) k_{\alpha,\mathfrak{p}}$ in the Iwasawa decomposition, and choose some element $\mathbf{e}_\alpha \in E^\times$ such that $\mathbf{e}_\alpha^{-1} a_{\alpha,\mathfrak{p}} \in \mathcal{O}^\times_{E_{\mathfrak{p}}}$. Let $\tilde{g}_\alpha = m(\mathbf{e}_\alpha^{-1}) g_\alpha$, and we have $\tilde{g}_{\alpha,\mathfrak{p}} = n(\tilde{b}_{\alpha,\mathfrak{p}})k_{\alpha,\mathfrak{p}}$ in the Iwasawa decomposition. Then $Z_{\phi_\alpha}(g_\alpha) = Z_{\phi_\alpha}(\tilde{g}_\alpha)$. As in 5.2.2, we have the series $\mathcal{Z}_{\phi_\alpha}(\tilde{g}_\alpha)_{0,K^{\mathfrak{p}}}$ for $\alpha = 1, 2$. Let $\mathcal{Z}_{\phi_\alpha}(g_\alpha) = \mathcal{Z}_{\phi_\alpha}(\tilde{g}_\alpha)_{0,K^{\mathfrak{p}}}$. The following is the main theorem of this chapter.

Theorem 5.3.3. *Suppose that $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for at least one place $v \in \mathsf{S}$, and $g_\alpha \in P'_v H'(\mathbb{A}_F^v)$ for $\alpha = 1, 2$. Let $\mathfrak{p}$ be a finite place that is not in S, $\mathcal{M}_{K;\mathfrak{p}^\circ} = \mathcal{M}_{0,K^{\mathfrak{p}}}$ the smooth local model introduced in 5.1.2, and $\mathcal{Z}_{\phi_\alpha}(g_\alpha)$ the series introduced above. Then we have*

1. *If $\mathfrak{p}$ is nonsplit in E, then*

$$E_{\mathfrak{p}}(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = -\operatorname{Vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ},$$

where

- *by definition,*
$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ} = \log q^2(\mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2));$$
- *$E_{\mathfrak{p}}(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ is defined in (2.15); and*
- *$\operatorname{Vol}(K)$ is the defined in Definition 4.3.3.*

2. *If $\mathfrak{p}$ is split in E, then*
$$\mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2) = 0.$$

Combining this with Theorem 4.3.4, we have the following corollary.

Corollary 5.3.4. *Assume that*

- *$\phi_\alpha = \phi_\infty^0 \phi_{\alpha,\mathrm{fin}}$ ($\alpha = 1, 2$) are decomposable as above;*
- *$\phi_{1,\mathsf{S}} \otimes \phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}_{\mathsf{S}}^2)_{\mathrm{reg}}$, i.e. $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for every $v \in \mathsf{S}$;*
- *$\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg},d_v}$ for every $v \in \mathsf{S}$ that is nonsplit and some $d_v \geq d_{\psi_v}$ (cf. 2.4.2);*
- *$g_\alpha \in e_{\mathsf{S}} H'(\mathbb{A}_F^{\mathsf{S}})$ ($\alpha = 1, 2$);*
- *the local model $\mathcal{M}_{K;\mathfrak{p}^\circ}$ is $\mathcal{M}_{0,K^{\mathfrak{p}}}$ for all finite places $\mathfrak{p}^\circ$ above $\mathfrak{p}$ that are not in S.*

Then

$$E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = -\operatorname{Vol}(K) \sum_{\substack{v^\circ | v \\ v \notin \mathsf{S}}} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ},$$

where the Green functions used in archimedean places are those defined in (4.55), which are not the admissible Green functions in the sense in 3.4.1.

Proof of Theorem 5.3.3. 1. By Lemma 5.1.6, the special fiber of $\mathcal{Z}_{\phi_\alpha}(g_\alpha)$ is located in the supersingular locus $[\mathcal{M}_{0,K^{\mathfrak{p}}}]_{\mathrm{ss}}$. If we denote by $[\mathcal{Z}_{\phi_\alpha}(g_\alpha)]^\wedge_{\mathrm{sp}}$ ($\alpha = 1, 2$) the completion along the special fiber, then

$$\mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2) = [\mathcal{Z}_{\phi_1}(g_1)]^\wedge_{\mathrm{sp}} \cdot [\mathcal{Z}_{\phi_2}(g_2)]^\wedge_{\mathrm{sp}} = [\mathcal{Z}_{\phi_1}(\tilde{g}_1)_{0,K^{\mathfrak{p}}}]^\wedge_{\mathrm{sp}} \cdot [\mathcal{Z}_{\phi_2}(\tilde{g}_2)_{0,K^{\mathfrak{p}}}]^\wedge_{\mathrm{sp}}. \tag{5.10}$$

Let $\breve{h}_i$ ($i = 1, \dots, l$) be a set of representatives of the double coset $[\mathcal{M}_{0,K^p}]_{\mathrm{ss}}(\mathbb{F}) \cong \breve{H}(\mathbb{Q})\backslash \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^p$. Then

$$(5.10) = \sum_{i=1}^{l} [\mathcal{Z}_{\omega_\chi(\breve{h}_i)\phi_1}(\tilde{g}_1)_{0,K^{\mathfrak{p}}}]^\wedge_s \cdot [\mathcal{Z}_{\omega_\chi(\breve{h}_i)\phi_2}(\tilde{g}_2)_{0,K^{\mathfrak{p}}}]^\wedge_s, \tag{5.11}$$

where we recall that $\left(\omega_\chi(\breve{h}_i)\phi_\alpha\right)(x) = \phi_\alpha(\breve{h}_i^{-1}x)$. By (5.8),

$$\begin{aligned}(5.11) &= \sum_{i=1}^{l} \left(\sum_{\breve{x}_1 \in \breve{V}} \psi_{\mathfrak{p}}(\tilde{b}_{1,\mathfrak{p}} T(\breve{x}_1)) \left(\breve{\phi}^0_{\mathfrak{p}} \otimes \left(\omega_\chi(\tilde{g}^{\mathfrak{p}}_1, \breve{h}_i)\phi^{\mathfrak{p}}_1 \right) \right) (\breve{x}_1) \mathscr{Z}(\breve{x}_1) \right) \\ &\qquad \cdot \left(\sum_{\breve{x}_2 \in \breve{V}} \psi_{\mathfrak{p}}(\tilde{b}_{2,\mathfrak{p}} T(\breve{x}_2)) \left(\breve{\phi}^0_{\mathfrak{p}} \otimes \left(\omega_\chi(\tilde{g}^{\mathfrak{p}}_2, \breve{h}_i)\phi^{\mathfrak{p}}_2 \right) \right) (\breve{x}_2) \mathscr{Z}(\breve{x}_2) \right) \\ &= \sum_{(\breve{x}_1, \breve{x}_2) \in \left(\breve{V} \cap \mathrm{Mor}((\mathbf{Y}, j_{\mathbf{Y}}), (\mathbf{X}, i_{\mathbf{X}}))\right)^2} \psi_{\mathfrak{p}}(\mathrm{tr}\, \tilde{b}_{\mathfrak{p}} T((\breve{x}_1, \breve{x}_2))) \\ &\qquad \sum_{i=1}^{l} \left(\omega''_\chi \left(\imath(\tilde{g}^{\mathfrak{p}}_1, \tilde{g}^{\mathfrak{p},\vee}_2)\right) \left(\phi^{\mathfrak{p}}_1 \otimes \phi^{\mathfrak{p}}_2\right)\right) (\breve{h}_i^{-1}(\breve{x}_1, \breve{x}_2)) \mathscr{Z}(\breve{x}_1) \cdot \mathscr{Z}(\breve{x}_2),\end{aligned} \tag{5.12}$$

where

$$\tilde{b}_{\mathfrak{p}} = \begin{pmatrix} \tilde{b}_{1,\mathfrak{p}} & 0 \\ 0 & \tilde{b}_{2,\mathfrak{p}} \end{pmatrix}.$$

Let $\breve{x}_T$ be a representative of the pairs $(\breve{x}_1, \breve{x}_2)$ such that $T((\breve{x}_1, \breve{x}_2)) = T$. Then

$$(5.12) = \sum_{T \in \mathrm{Her}_2(E_{\mathfrak{p}^\circ}) \cap \mathrm{GL}_2(\mathcal{O}_{E_{\mathfrak{p}^\circ}})} \psi_{\mathfrak{p}}(\mathrm{tr}\, \tilde{b}_{\mathfrak{p}} T) \sum_{\breve{h} \in \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}} \left(\omega''_\chi \left(\imath(\tilde{g}^{\mathfrak{p}}_1, \tilde{g}^{\mathfrak{p},\vee}_2)\right) \left(\phi^{\mathfrak{p}}_1 \otimes \phi^{\mathfrak{p}}_2\right)\right) (\breve{h}^{-1}\breve{x}_T) H_{\mathfrak{p}}(T).$$

By Theorem 5.2.3, Corollary 5.3.2, and following the same steps in the proof of Theorem 4.3.4, we obtain that

$$-\operatorname{Vol}(K)\log q^2(\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{Z}_{\phi_2}(g_2)) = E_{\mathfrak{p}}(0,\imath(\tilde{g}_1,\tilde{g}_2^\vee),\phi_1\otimes\phi_2). \tag{5.13}$$

Let

$$\mathbf{e} = \begin{pmatrix} \mathbf{e}_1 & \\ & \mathbf{e}_2 \end{pmatrix}.$$

By definition,

$$\begin{aligned}
(5.13) &= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W'_T(0,\imath(\tilde{g}_{1,\mathfrak{p}},\tilde{g}_{2,\mathfrak{p}}^\vee),\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}})\prod_{v\neq\mathfrak{p}} W_T(0,\imath(\tilde{g}_{1,v},\tilde{g}_{2,v}^\vee),\phi_{1,v}\otimes\phi_{2,v})\\
&= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W'_{\mathbf{e}^\tau T\mathbf{e}}(0,\imath(g_{1,\mathfrak{p}},g_{2,\mathfrak{p}}^\vee),\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}})\prod_{v\neq\mathfrak{p}} W_{\mathbf{e}^\tau T\mathbf{e}}(0,\imath(g_{1,v},g_{2,v}^\vee),\phi_{1,v}\otimes\phi_{2,v})\\
&= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W'_T(0,\imath(g_{1,\mathfrak{p}},g_{2,\mathfrak{p}}^\vee),\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}})\prod_{v\neq\mathfrak{p}} W_T(0,\imath(g_{1,v},g_{2,v}^\vee),\phi_{1,v}\otimes\phi_{2,v})\\
&= E_{\mathfrak{p}}(0,\imath(g_1,g_2^\vee),\phi_1\otimes\phi_2).
\end{aligned}$$

Therefore, (1) is proved.

2. It will be proved in a more general context in Lemma 6.1.1.

□

Chapter 6

Comparison at finite places: bad reduction

In this chapter, we discuss the local height pairing $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}$ on certain model $\mathcal{M}_K$ at every bad place $\mathfrak{p} \in \mathsf{S}$ when $n = 1$. We will assume that

- $\phi_\alpha = \phi^0_\infty \phi_{\alpha,\mathrm{fin}}$ $(\alpha = 1, 2)$ are decomposable;
- $\phi_{\alpha,\mathsf{S}} \in \mathcal{S}(\mathbb{V}_\mathsf{S})_{\mathrm{reg}}$ $(\alpha = 1, 2)$;
- $\phi_{1,\mathsf{S}} \otimes \phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}^2_\mathsf{S})_{\mathrm{reg}}$;

and $g_\alpha \in e_\mathsf{S} H'(\mathbb{A}^\mathsf{S}_F)$ for $\alpha = 1, 2$.

In 6.1, we discuss the contribution of the local height pairing at a finite place $\mathfrak{p}$ in S that is split in E. In 6.2, we discuss the contribution of the local height pairing at a finite place $\mathfrak{p}$ in S that is nonsplit in E and such that $\epsilon(V_\mathfrak{p}) = 1$. In 6.3, we discuss the contribution of the local height pairing at a finite place $\mathfrak{p}$ in S that is nonsplit in E and such that $\epsilon(V_\mathfrak{p}) = -1$.

6.1 Split case

In this section, we discuss the contribution of the local height pairing at a finite place $\mathfrak{p}$ in S that is split in E. Let $\mathfrak{p}^\circ$ be a place in $\Sigma^\circ_{\mathrm{fin}}$ lying over $\mathfrak{p}$.

6.1.1 Integral models and ordinary reduction

Let $K = K_\mathfrak{p}K^\mathfrak{p}$ be an open compact subgroup of $H(\mathbb{A}_{\mathrm{fin}})$ with $K^\mathfrak{p}$ sufficiently small and $K_\mathfrak{p} = K_{\mathfrak{p},n}$ for $n \geq 0$. Therefore, $M_K = M_{n,K^\mathfrak{p}}$. In 5.1.2, we construct a smooth integral model $\mathcal{M}_{0,K^\mathfrak{p}}$ for $M_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$ on which there is a p-divisible group $\mathcal{X}$. Then $\mathcal{X}^\flat \to \mathcal{M}_{0,K^\mathfrak{p}}$ is an $\mathcal{O}_{F_\mathfrak{p}}$-module of dimension 1 and height 2. Recall that a Drinfeld ϖ^n-structure for an $\mathcal{O}_{F_\mathfrak{p}}$-module X of dimension 1 and height 2 over an $\mathcal{O}_{F_\mathfrak{p}}$-scheme S is an $\mathcal{O}_{F_\mathfrak{p}}$-homomorphism

$$\alpha_n : (\mathcal{O}_{F_\mathfrak{p}}/\varpi^n\mathcal{O}_{F_\mathfrak{p}})^2 \to X[\varpi^n](S)$$

such that the image forms a full set of sections of $X[\varpi^n]$ in the sense of [KM1985, 1.8]. Let $\mathcal{M}_{n,K^\mathfrak{p}} = \mathcal{M}_{0,K^\mathfrak{p}}(n)$ be the universal scheme over $\mathcal{M}_{0,K^\mathfrak{p}}$ of the Drinfeld ϖ^n-structure (*cf.* [HT2001, Lemma II.2.1]). Then $\mathcal{M}_{n,K^\mathfrak{p}}$ is regular and finite over $\mathcal{M}_{0,K^\mathfrak{p}}$, whose generic fiber is $M_{n,K^\mathfrak{p}}$. Let $M'_{n,K^\mathfrak{p}} = M_{n,K^\mathfrak{p}} \times_{F_\mathfrak{p}} F^n_\mathfrak{p}$ be the base change, and $\mathcal{M}'_{n,K^\mathfrak{p}}$ the normalization of $\mathcal{M}_{n,K^\mathfrak{p}} \times_{\mathcal{O}_{F_\mathfrak{p}}} \mathcal{O}_{F^n_\mathfrak{p}}$ that is regular. We denote by $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{ord}}$ the ordinary locus as well as the smooth locus, which is an open subscheme, of the special fiber $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{sp}}$.

The set of connected components of $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{sp}}$ corresponds canonically to the set of geometric connected components of $M_{n,K^\mathfrak{p}}$, hence to $E^{\times,1}\backslash\mathbb{A}_E^{\times,1}/\nu(K)$. The set of irreducible components on each connected component of $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{sp}}$, that is, the Igusa curves, corresponds to the set $\mathbb{P}(V_\mathfrak{p})/K_{\mathfrak{p},n}$. Here, $\mathbb{P}(V_\mathfrak{p})$ is the set of all rank 1 $E_\mathfrak{p} \cong F_\mathfrak{p} \oplus F_\mathfrak{p}$-submodules in $V_\mathfrak{p}$, where $\mathrm{U}(V_\mathfrak{p})$ acts from right by $l.h = h^{-1}l$ for $l \in \mathbb{P}(V_\mathfrak{p})$ and $h \in \mathrm{U}(V_\mathfrak{p})$. Together, the set of irreducible components of $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{sp}}$ can be identified with

$$\mathrm{Ig}_{n,K^\mathfrak{p}} = \mathbb{P}(V_\mathfrak{p})/K_{\mathfrak{p},n} \times \left(E^{\times,1}\backslash\mathbb{A}_E^{\times,1}/\nu(K)\right).$$

We consider special cycles. We keep the same notations for the base change of special cycles $Z(x)_K$ and the generating series $Z_{\phi_\alpha}(g_\alpha)$ on $M'_{n,K^\mathfrak{p}}$. Let $\mathcal{Z}(x)_K$ (resp. $\mathcal{Z}_{\phi_\alpha}(g_\alpha)$) be the Zariski closure of $Z(x)_K$ (resp. $Z_{\phi_\alpha}(g_\alpha)$) in $\mathcal{M}'_{n,K^\mathfrak{p}}$. Since $\mathfrak{p}$ is split in E, the special fiber $[\mathcal{Z}_{\phi_\alpha}(g_\alpha)]_{\mathrm{sp}}$ is contained in the ordinary locus $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{ord}}$. Let $\mathbb{P}(V)^+$ be the set of totally positive definite E-lines in V. Then the set of geometric special points of $M_{n,K^{\mathfrak{p}}}$ (also of $M_{n,K^{\mathfrak{p}}}$ and $M'_{n,K^{\mathfrak{p}}}$) can be identified with

$$\mathrm{Sp}_K = H(\mathbb{Q})\backslash\mathbb{P}(V)^+ \times H(\mathbb{A}_{\mathrm{fin}})/K = \coprod_{l\in H(\mathbb{Q})\backslash\mathbb{P}(V)^+} H_l(\mathbb{Q})\backslash H(\mathbb{A}_{\mathrm{fin}})/K,$$

and the set $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{ord}}(\mathbb{F})$ can be identified with

$$\coprod_{l\in H(\mathbb{Q})\backslash\mathbb{P}(V)} H_l(\mathbb{Q})\backslash\left((N_l\backslash \mathrm{U}(V_{\mathfrak{p}})/K_{\mathfrak{p},n}) \times H^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}\right),$$

where $N_l \subset \mathrm{U}(V_{\mathfrak{p}})$ is the unipotent radical of the parabolic subgroup stabilizing l. The reduction map

$$\mathrm{Sp}_K \to [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{ord}}(\mathbb{F}) \to \mathrm{Ig}_{n,K^{\mathfrak{p}}} \tag{6.1}$$

is given by

$$(l,h) \mapsto (l, h_{\mathfrak{p}}, h^{\mathfrak{p}}) \mapsto (h_{\mathfrak{p}}^{-1}l, \nu(h_{\mathfrak{p}}h^{\mathfrak{p}}))$$

(*cf.* [Zha2001b, 5.4] for a discussion).

6.1.2 Coherence for intersection numbers

We compute the local height pairing on the integral model $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. Write $\widehat{Z}_{\phi_\alpha} = \mathcal{Z}_{\phi_\alpha}(g_\alpha) + \mathcal{V}_{\phi_\alpha}(g_\alpha)$ for some divisor $\mathcal{V}_{\phi_\alpha}(g_\alpha)$ supported on the special fiber as in 3.4.3. We have

$$\begin{aligned}
&(\log q)^{-1}\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ} \\
&= (\mathcal{Z}_{\phi_1}(g_1) + \mathcal{V}_{\phi_1}(g_1)) \cdot (\mathcal{Z}_{\phi_2}(g_2) + \mathcal{V}_{\phi_2}(g_2) - E(g_2,\phi_2)\omega_K + E(g_2,\phi_2)\omega_K) \\
&= \mathcal{Z}_{\phi_1}(g_1) \cdot (\mathcal{Z}_{\phi_2}(g_2) + \mathcal{V}_{\phi_2}(g_2) - E(g_2,\phi_2)\omega_K) + E(g_2,\phi_2)(\mathcal{Z}_{\phi_1}(g_1) + \mathcal{V}_{\phi_1}(g_1)) \cdot \omega_K) \\
&= \mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2) + \mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{V}_{\phi_2}(g_2) + E(g_2,\phi_2)\mathcal{V}_{\phi_1}(g_1) \cdot \omega_K,
\end{aligned} \tag{6.2}$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$. First, we have the following simple lemma.

Lemma 6.1.1. *Under the weaker hypotheses that $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}} = \mathcal{S}(V_v^2)_{\mathrm{reg}}$ and $g_\alpha \in P'_v H'(\mathbb{A}_F^v)$ $(\alpha = 1,2)$ for some finite place v other than $\mathfrak{p}$, $\mathcal{Z}_{\phi_1}(g_1)$ and $\mathcal{Z}_{\phi_2}(g_2)$ do not intersect.*

Proof. It follows immediately from the first map in (6.1). □

Second, we define a function $\nu(\bullet, \phi_2, g_2)$ on $V_{\mathfrak{p}} - \{0\}$ in the following way. For any nonzero $x \in V_{\mathfrak{p}}$, let l_x be the line spanned by x, which is an element in $\mathbb{P}(V_{\mathfrak{p}})$. Set $\nu(x, \phi_2, g_2)$ to be the coefficient of the geometric irreducible component represented by $(l_x, 1)$ in $\mathrm{Ig}_{n,K^{\mathfrak{p}}}$ in $\mathcal{V}_{\phi_2}(g_2)$. It is a locally constant function, and

$$\nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) = \frac{\mathrm{Vol}(\det K)}{\mathrm{Vol}(K)} \phi_{1,\mathfrak{p}} \otimes \nu(\bullet, \phi_2, g_2)$$

extends to a function in $\mathcal{S}(V_{\mathfrak{p}})$ such that $\nu(0, \phi_{1,\mathfrak{p}}; \phi_2, g_2) = 0$ since $\phi_{1,\mathfrak{p}}(0) = 0$. Then the intersection number

$$\begin{aligned} \mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{V}_{\phi_2}(g_2) &= \sum_{x \in K \backslash V \otimes \mathbb{A}_{\mathrm{fin}}} (\omega_\chi(g_1)\phi_1)(x) \mathcal{Z}(x)_K \cdot \mathcal{V}_{\phi_2}(g_2) && (6.3) \\ &= \sum_{\substack{x \in K \backslash V \otimes \mathbb{A}_{\mathrm{fin}} \\ T(x) \in F^+}} \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(K \cap H(\mathbb{A}_{\mathrm{fin}})_x)} \left(\nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \left(\omega_\chi(g_1^{\mathfrak{p}}) \phi_1^{\mathfrak{p}} \right) \right)(x), && (6.4) \end{aligned}$$

since $g_1 \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$. If we let

$$E(s, g, \nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}}) = \sum_{\gamma \in P'(F) \backslash H'(F)} \left(\omega_\chi(\gamma g) \left(\nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}} \right) \right)(0) \lambda_{P'}(\gamma g)^{s - \frac{1}{2}}$$

be an Eisenstein series that is holomorphic at $s = \frac{1}{2}$, then we have

$$(6.3) = E(s, g_1, \nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}}) \mid_{s=\frac{1}{2}} - W_0(\frac{1}{2}, g_1, \nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}})$$

by the standard Siegel–Weil argument, which is similar to Proposition 3.4.1. For simplicity, we write

$$E_{(\mathfrak{p}^\circ)}(\phi_1, g_1; \phi_2, g_2) = \log q \left(E(s, g_1, \nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}}) \mid_{s=\frac{1}{2}} - W_0(\frac{1}{2}, g_1, \nu(\bullet, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}}) \right).$$

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) = \log q \left(\mathcal{V}_{\phi_1}(g_1) \cdot \omega_K \right).$$

We have the following proposition.

Proposition 6.1.2. *Suppose that $\phi_{\alpha,\mathsf{S}} \in \mathcal{S}(\mathbb{V}_{\mathsf{S}})_{\mathrm{reg}}$ $(\alpha = 1, 2)$, $\phi_{1,\mathsf{S}} \otimes \phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}_{\mathsf{S}}^2)_{\mathrm{reg}}$, and $g_\alpha \in e_{\mathsf{S}} H'(\mathbb{A}_F^{\mathsf{S}})$ for $\alpha = 1, 2$. Then*

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ} = E_{(\mathfrak{p}^\circ)}(\phi_1, g_1; \phi_2, g_2) + A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) E(g_2, \phi_2).$$

6.2 Quasi-split case

In this section, we discuss the contribution of the local height pairing at a finite place $\mathfrak{p}$ in S that is nonsplit in E and such that $\epsilon(V_\mathfrak{p}) = 1$. Let $\mathfrak{p}^\circ$ be the only place in $\Sigma^\circ_{\mathrm{fin}}$ lying over $\mathfrak{p}$. We identify $V_\mathfrak{p}$ with $\mathrm{Mat}_2(F_\mathfrak{p})$ and the lattice Λ^+ with $\mathrm{Mat}_2(\mathcal{O}_{F_\mathfrak{p}})$.

6.2.1 Integral models and supersingular reduction

Let $K = K_\mathfrak{p} K^\mathfrak{p}$ be an open compact subgroup of $H(\mathbb{A}_{\mathrm{fin}})$ with $K^\mathfrak{p}$ sufficiently small and $K_\mathfrak{p} = K_{\mathfrak{p},n}$ for n large such that $F^n_\mathfrak{p}$ contains $E_{\mathfrak{p}^\circ}$. Therefore, $M_K = M_{n,K^\mathfrak{p}}$. In 5.1.2, we construct a smooth integral model $\mathcal{M}_{0,K^\mathfrak{p}}$ for $M_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$ on which there is a p-divisible group $\mathcal{X}$. Let $\mathcal{M}_{n,K^\mathfrak{p}}$ be the normalization of $\mathcal{M}_{0,K^\mathfrak{p}}$ in $M_{n,K^\mathfrak{p}}$, which is regular and finite over $\mathcal{M}_{0,K^\mathfrak{p}}$. Consider the base change $M'_{n,K^\mathfrak{p}} = M_{n,K^\mathfrak{p}} \times_{E_{\mathfrak{p}^\circ}} F^n_\mathfrak{p}$. Let $\mathcal{M}'_{n,K^\mathfrak{p}}$ be the normalization of $\mathcal{M}_{n,K^\mathfrak{p}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_\mathfrak{p}}$ that is a regular model of $M'_{n,K^\mathfrak{p}}$. We have the following description of the supersingular locus

$$[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{ss}}(\mathbb{F}) \cong \breve{H}(\mathbb{Q})\backslash \left(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^\mathfrak{p}_{\mathrm{fin}}/K^\mathfrak{p}\right),$$

where $\breve{H}(\mathbb{Q})$ acts on the first factor through the determinant.

We denote by $\mathcal{X}^{\mathrm{univ}}$ the p-divisible group over $\mathscr{N} \cong \mathrm{Spf}\, R_{F_\mathfrak{p},2}$. Let $R_{F_\mathfrak{p},2,n}$ (*cf.* [HT2001, Lemma II.2.2]) be such that $\mathrm{Spec}\, R_{F_\mathfrak{p},2,n} = (\mathrm{Spec}\, R_{F_\mathfrak{p},2})(n)$ is the universal scheme of the Drinfeld ϖ^n-structure for $\mathcal{X}^{\mathrm{univ},\flat}$ that is in fact defined over $\mathrm{Spec}\, R_{F_\mathfrak{p},2}$. Let $\mathrm{Spec}\, R'_n$ be the normalization of $\mathrm{Spec}\left(R_{F_\mathfrak{p},2} \otimes_{\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} \mathcal{O}_{\widehat{F^n_\mathfrak{p}}}\right)$ in $\mathrm{Spec}\left(R_{F_\mathfrak{p},2,n} \otimes_{\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} \widehat{F^n_\mathfrak{p}}\right)$. The set of connected components of $\mathrm{Spec}\, R'_n$ is parameterized by the coset $\mathcal{O}^\times_{F_\mathfrak{p}}/\nu(K^\dagger_{\mathfrak{p},n})$. Let $\mathscr{N}'_n$ be the neutral connected component of $\mathrm{Spf}\, R'_n$, which is finite over $\mathscr{N}' \times_{\mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}}} \mathcal{O}_{\widehat{F^n_\mathfrak{p}}}$. Its generic fiber $\mathscr{N}'_{n,\eta}$ is Galois over $\mathscr{N}'_\eta = \mathrm{Spf}\, R_{F_\mathfrak{p},2} \otimes_{\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} \widehat{F^n_\mathfrak{p}}$ with the Galois group

$$\overline{K}_{\mathfrak{p},n} := \ker\left(\mathrm{GL}(\Lambda^{+,\flat}/\varpi^n\Lambda^{+,\flat}) \xrightarrow{\wedge^2} \mathrm{GL}(\mathcal{O}_{F_\mathfrak{p}}/\varpi^n\mathcal{O}_{F_\mathfrak{p}})\right),$$

which fits into the following exact sequence

$$1 \longrightarrow \overline{K}_{\mathfrak{p},n} \longrightarrow K_{\mathfrak{p},0}/K_{\mathfrak{p},n} \xrightarrow{\overline{\nu}} E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n}) \longrightarrow 1\,.$$

We define the universal p-divisible group $\mathcal{X}'_n$ over $\mathscr{N}'_n$ by the following Cartesian diagram

$$\begin{tikzcd}
\mathcal{X}'_n \arrow[r] \arrow[d] & \mathscr{N}'_n \arrow[d] \\
\mathcal{X}^{\mathrm{univ}} \times_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}} \arrow[r] & \mathscr{N} \times_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}.
\end{tikzcd}$$

Moreover, we have the universal Drinfeld ϖ^n-structure

$$\alpha'_{n,\eta} : \Lambda^{+,\flat}/\varpi^n\Lambda^{+,\flat} \to \mathcal{X}'^{\flat}_{n,\eta}[\varpi^n](\mathscr{N}'_{n,\eta})$$

for $\mathcal{X}'^{\flat}_{n,\eta}$. In particular, we have the following p-adic uniformization for the completion of $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ along the supersingular locus

$$[\mathcal{M}'_{n,K^{\mathfrak{p}}}]^{\wedge}_{\mathrm{ss}} \cong \breve{H}(\mathbb{Q})\backslash \left(\mathscr{N}'_n \times E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}\right).$$

In 5.1.5, we have defined a 1-dimensional closed subscheme $\mathcal{Z}(x_0)_{0,K^{\mathfrak{p}}}$ of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ whose generic fiber is $Z(x_0)_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, for a $K_{\mathfrak{p},0}K^{\mathfrak{p}}$-orbit $K_{\mathfrak{p},0}K^{\mathfrak{p}}x_0$ in $V(\mathbb{A}_{F,\mathrm{fin}})$ that is admissible. Define $\mathcal{Z}'(x_0)_{n,K^{\mathfrak{p}}}$ by the following Cartesian diagram

$$\begin{tikzcd}
\mathcal{Z}'(x_0)_{n,K^{\mathfrak{p}}} \arrow[r] \arrow[dd] & \mathcal{M}'_{n,K^{\mathfrak{p}}} \arrow[d] \\
 & \mathcal{M}_{n,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_{\mathfrak{p}}} \arrow[d] \\
\mathcal{Z}(x_0)_{0,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_{\mathfrak{p}}} \arrow[r] & \mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_{\mathfrak{p}}}.
\end{tikzcd}$$

It is easy to see that the $K_{\mathfrak{p},n}K^{\mathfrak{p}}$-orbits inside $K_{\mathfrak{p},0}K^{\mathfrak{p}}x_0$ are parameterized by the finite group $\overline{K}_{\mathfrak{p},n}$. For every such orbit $K_{\mathfrak{p},n}K^{\mathfrak{p}}x$, we define $\mathcal{Z}(x)_{n,K^{\mathfrak{p}}}$ to be the union of irreducible components of $\mathcal{Z}'(x_0)_{n,K^{\mathfrak{p}}}$ whose generic fiber contributes to the special divisor $Z(x)_{n,K^{\mathfrak{p}}} \times_{E_{\mathfrak{p}^\circ}} F^n_{\mathfrak{p}}$ on the generic fiber $M'_{n,K^{\mathfrak{p}}}$.

In 5.2.2, we have defined $\mathscr{Z}(\breve{x})$ that is a formal divisor of $\mathscr{N}$, for $\breve{x} \in \mathrm{Mor}\left((\mathbf{Y}, j_{\mathbf{Y}}), (\mathbf{X}, i_{\mathbf{X}})\right)_{\mathrm{reg}}$.

Define $\mathscr{Z}'(\breve{x})_n$ by the following Cartesian diagram

$$\begin{tikzcd}
\mathscr{Z}'(\breve{x})_n \arrow[r] \arrow[d] & \mathscr{N}'_n \arrow[d] \\
\mathscr{Z}(\breve{x}) \times_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}} \arrow[r] & \mathscr{N} \times_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}.
\end{tikzcd}$$

If we denote by $\widehat{\mathcal{Y}}$ the completion of $\mathcal{Y}$ along the special fiber, we have a universal homomorphism

$$[\breve{x}] : \widehat{\mathcal{Y}} \times_{\mathrm{Spf}\, \mathcal{O}_{\widehat{F^\circ_{\mathfrak{p}}}}} \mathscr{Z}'(\breve{x})_n \to \mathcal{X}'_n \mid \mathscr{Z}'(\breve{x})_n$$

of p-divisible groups over the formal scheme $\mathscr{Z}'(\breve{x})_n$. Pick up any geometric point z of characteristic 0 on $\mathscr{Z}'(\breve{x})_n$ and choose a similitude $\mathcal{X}'_{n,z} \cong \Lambda^+$ of $\mathcal{O}_{F_{\mathfrak{p}}}$-symplectic modules that induces the universal Drinfeld ϖ^n-structure $\alpha'_{n,\eta}$ at the point s. Then $[\breve{x}]_*(\mathbf{x}_{\mathfrak{p}})$ defines a $K^\dagger_{\mathfrak{p},n}$-orbit in $\Lambda^+ \subset V_{\mathfrak{p}}$ such that $\mathrm{Ht}(\breve{x}) = \mathrm{Ht}(x)$ for every x in the orbit. Here,

- $\mathbf{x}_{\mathfrak{p}}$ is induced from $\mathbf{x}$ in the fixed unitary data;
- $\mathrm{Ht}(\breve{x})$ is the integer h (≥ 0) such that $\frac{1}{2}\operatorname{Tr}_{E_{\mathfrak{p}^\circ}/F_{\mathfrak{p}}}(\breve{x}, \breve{x})' \in \varpi^h \mathcal{O}^\times_{F_{\mathfrak{p}}}$;
- $\mathrm{Ht}(x)$ is the largest integer h (≥ 0) such that $\det x \in \varpi^h \mathcal{O}^\times_{F_{\mathfrak{p}}}$.

Therefore, we have the following decomposition

$$\mathscr{Z}'(\breve{x})_n = \bigcup_{\substack{x \in K^\dagger_{\mathfrak{p},n} \backslash \Lambda^+ \\ \mathrm{Ht}(x) = \mathrm{Ht}(\breve{x})}} \mathscr{Z}(\breve{x}, x)_n$$

into (finitely many) formal divisors of $\mathscr{N}'_n$.

Let $\phi = \phi^0_\infty \otimes (\otimes_{v \in \Sigma_{\mathrm{fin}}} \phi_v)$ such that

- $\phi^{\mathfrak{p}}_{\mathrm{fin}} := \otimes_{v \in \Sigma_{\mathrm{fin}} - \{\mathfrak{p}\}} \phi_v$ is in $\mathcal{S}(V(\mathbb{A}^{\mathfrak{p}}_{F,\mathrm{fin}}))^{K^{\mathfrak{p}}}$;
- $\phi_{\mathfrak{p}}$ is in $\mathcal{S}(V_{\mathfrak{p}})_{\mathrm{reg}} \cap \mathcal{S}(V_{\mathfrak{p}})^{K^\dagger_{\mathfrak{p},n}}$, and its support is contained in Λ^+.

In particular, $\phi(0) = 0$. Consider the generating series

$$Z_\phi(e_{\mathfrak{p}} g^{\mathfrak{p}}) = \sum_{x \in K_{\mathfrak{p},n} K^{\mathfrak{p}} \backslash V(\mathbb{A}_{F,\mathrm{fin}}) - \{0\}} \left(\phi_{\mathfrak{p}} \otimes (\omega_\chi(g^{\mathfrak{p}}) \phi^{\mathfrak{p}})\right)(x) Z(x)_{K_{\mathfrak{p},n} K^{\mathfrak{p}}}.$$

Define

$$\mathcal{Z}_\phi(e_\mathfrak{p} g^\mathfrak{p})_{n,K^\mathfrak{p}} = \sum_{x \in K_{\mathfrak{p},n}K^\mathfrak{p} \backslash V(\mathbb{A}_{F,\mathrm{fin}})-\{0\}} \left(\phi_\mathfrak{p} \otimes (\omega_\chi(g^\mathfrak{p})\phi^\mathfrak{p})\right)(x)\mathcal{Z}(x)_{n,K^\mathfrak{p}},$$

whose generic fiber is the base change of $Z_\phi(e_\mathfrak{p} g^\mathfrak{p})$ to $F^n_\mathfrak{p}$. Moreover, its completion at the closed point s_n of $\mathscr{N}'_n$ is

$$[\mathcal{Z}_\phi(e_\mathfrak{p} g^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{s_n} = \sum_{\breve{x}\in\breve{V}} \sum_{\substack{x\in K_{\mathfrak{p},n}\backslash\Lambda^+ \\ (x,x)=(\breve{x},\breve{x})'}} \left(\omega_\chi(g^\mathfrak{p})\phi^\mathfrak{p}\right)(\breve{x})\phi_\mathfrak{p}(x)\mathscr{Z}(\breve{x},x)_n. \tag{6.5}$$

6.2.2 Coherence for intersection numbers

Define the subset

$$\mathrm{Red}^{\mathrm{qs}}_n = \{(x_1,x_2;\breve{x}_1,\breve{x}_2)\} \subset \left(\Lambda^+\right)^2 \times \mathrm{Mor}\left((\mathbf{Y}, j_\mathbf{Y}),(\mathbf{X}, i_\mathbf{X})\right)^2_{\mathrm{reg}}$$

by the conditions that

- $\mathrm{Ht}(x_\alpha) = \mathrm{Ht}(\breve{x}_\alpha)$ for $\alpha = 1, 2$;
- $K^\dagger_{\mathfrak{p},n}x_1$ and $K^\dagger_{\mathfrak{p},n}x_2$ are linearly independent.

Then by definition, for $(x_1,x_2;\breve{x}_1,\breve{x}_2) \in \mathrm{Red}^{\mathrm{qs}}_n$, the formal divisors $\mathscr{Z}(\breve{x}_1,x_1)_n$ and $\mathscr{Z}(\breve{x}_2,x_2)_n$ will intersect properly. We let

$$m(x_1,x_2;\breve{x}_1,\breve{x}_2) = \mathscr{Z}(\breve{x}_1,x_1)_n \cdot \mathscr{Z}(\breve{x}_2,x_2)_n$$

be the intersection multiplicity, which is a well-defined continuous function on $\mathrm{Red}^{\mathrm{qs}}_n$. The following lemma is straightforward.

Lemma 6.2.1. *Suppose that for $\alpha = 1, 2$, $\phi_{\alpha,\mathfrak{p}} \in \mathcal{S}(V_\mathfrak{p})_{\mathrm{reg}} \cap \mathcal{S}(V_\mathfrak{p})^{K^\dagger_{\mathfrak{p},n}}$ whose support is contained in Λ^+ and such that $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}$ is in $\mathcal{S}(V^2_\mathfrak{p})_{\mathrm{reg}}$. Then the following function*

$$\mu(\breve{x}_1,\breve{x}_2;\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}) = \sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^+ \\ (x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}} \sum_{\substack{x_2\in K_{\mathfrak{p},n}\backslash\Lambda^+ \\ (x_2,x_2)=(\breve{x}_2,\breve{x}_2)'}} \left(\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}\right)(x_1,x_2)m(x_1,x_2;\breve{x}_1,\breve{x}_2),$$

which is a priori *defined on* $\mathrm{Mor}\left((\mathbf{Y}, j_\mathbf{Y}),(\mathbf{X}, i_\mathbf{X})\right)^2_{\mathrm{reg}}$*, is a Schwartz function in $\mathcal{S}(\breve{V}^2_\mathfrak{p})$ via extension by zero.*

Let $h_{\mathfrak{p},j} \in K_{\mathfrak{p},0}$ $(j = 1, \ldots, m)$ be a set of representatives of the coset $\overline{K}_{\mathfrak{p},n} \backslash K_{\mathfrak{p},0} / K_{\mathfrak{p},n}$, and define

$$\breve{\mu}(\breve{x}_1, \breve{x}_2; \phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}) = \sum_{j=1}^{m} \mu\left(\breve{x}_1, \breve{x}_2; \omega''_\chi(h_{\mathfrak{p},j})\left(\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}\right)\right),$$

whose value does not depend on the representatives we choose.

We return to our original assumption for this chapter that $\phi_{\alpha,\mathsf{S}} \in \mathcal{S}(\mathbb{V}_\mathsf{S})_{\mathrm{reg}}$ $(\alpha = 1, 2)$, $\phi_{1,\mathsf{S}} \otimes \phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}_\mathsf{S}^2)_{\mathrm{reg}}$, and $g_\alpha \in e_\mathsf{S} H'(\mathbb{A}_F^\mathsf{S})$ for $\alpha = 1, 2$. Choose some element $\mathbf{e}_\alpha \in E^\times$ such that $\omega_\chi(m(\mathbf{e}_\alpha))\phi_{\alpha,\mathfrak{p}}$ is supported on Λ^+ for $\alpha = 1, 2$. We have

$$\begin{aligned} Z_{\phi_\alpha}(g_\alpha) &= \sum_{x_\alpha \in K \backslash \mathbb{V}_{\mathrm{fin}}} \left(\omega_\chi(g_\alpha)\phi_\alpha\right)(x_\alpha) Z(x_\alpha)_K \\ &= \sum_{x_\alpha \in K \backslash \mathbb{V}_{\mathrm{fin}}} \left(\omega_\chi(g_\alpha)\phi_\alpha\right)(x_\alpha \mathbf{e}_\alpha) Z(x_\alpha \mathbf{e}_\alpha)_K \\ &= \sum_{x_\alpha \in K \backslash \mathbb{V}_{\mathrm{fin}}} \left(\omega_\chi(g_\alpha)\phi_\alpha\right)(x_\alpha \mathbf{e}_\alpha) Z(x_\alpha)_K. \end{aligned}$$

Therefore, we can add one more assumption that $\phi_{\alpha,\mathfrak{p}}$ is in $\mathcal{S}(V_\mathfrak{p})_{\mathrm{reg}} \cap \mathcal{S}(V_\mathfrak{p})^{K^\dagger_{\mathfrak{p},n}}$, and its support is contained in Λ^+. We let $\mathcal{Z}_{\phi_\alpha}(g_\alpha) = \mathcal{Z}_\phi(e_\mathfrak{p} g_\alpha^\mathfrak{p})_{n,K^\mathfrak{p}}$ and denote by $[\mathcal{Z}_\phi(g_\alpha)]^\wedge_{\mathrm{sp}}$ its completion along the special fiber that is contained in the supersingular locus $[\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathrm{ss}}$. We have a similar decomposition as (6.2).

First, we consider

$$\mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2) = [\mathcal{Z}_{\phi_1}(g_1)]^\wedge_{\mathrm{sp}} \cdot [\mathcal{Z}_{\phi_2}(g_2)]^\wedge_{\mathrm{sp}} = [\mathcal{Z}_{\phi_1}(e_\mathfrak{p} g_1^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{\mathrm{sp}} \cdot [\mathcal{Z}_{\phi_2}(e_\mathfrak{p} g_2^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{\mathrm{sp}}. \tag{6.6}$$

Let $\breve{h}_i^\mathfrak{p} \in \breve{H}_{\mathrm{fin}}^\mathfrak{p}$ $(i = 1, \ldots, l)$ be a set of representatives of the double coset $\breve{H}(\mathbb{Q}) \backslash \breve{H}_{\mathrm{fin}}^\mathfrak{p} / K^\mathfrak{p}$.

Then

$$(6.6) = \sum_{i=1}^{l} \sum_{j=1}^{m} [\mathcal{Z}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^\mathfrak{p})\phi_1}(e_\mathfrak{p} g_1^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{s_n} \cdot [\mathcal{Z}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^\mathfrak{p})\phi_2}(e_\mathfrak{p} g_2^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{s_n}. \tag{6.7}$$

By (6.5),

$$
\begin{aligned}
(6.7) = \sum_{i=1}^{l}\sum_{j=1}^{m}&\left(\sum_{\breve{x}_1\in\breve{V}}\sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^+\\(x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}}\left(\omega_\chi(g_1^{\mathfrak{p}})\phi_1^{\mathfrak{p}}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_1)\phi_{1,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_1)\mathscr{Z}(\breve{x}_1,x_1)_n\right)\\
&\cdot\left(\sum_{\breve{x}_2\in\breve{V}}\sum_{\substack{x_2\in K_{\mathfrak{p},n}\backslash\Lambda^+\\(x_2,x_2)=(\breve{x}_2,\breve{x}_2)'}}\left(\omega_\chi(g_2^{\mathfrak{p}})\phi_2^{\mathfrak{p}}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_2)\phi_{2,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_2)\mathscr{Z}(\breve{x}_2,x_2)_n\right)\\
=\sum_{i=1}^{l}&\sum_{(\breve{x}_1,\breve{x}_2)\in\breve{V}^2}\left(\omega''_\chi\left(\imath(g_1^{\mathfrak{p}},g_2^{\mathfrak{p},\vee})\right)\left(\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}\right)\right)(\breve{h}_i^{\mathfrak{p},-1}(\breve{x}_1,\breve{x}_2))\breve{\mu}(\breve{x}_1,\breve{x}_2;\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}). \qquad (6.8)
\end{aligned}
$$

We define

$$
\Phi^{\mathrm{hor}}=\sum_{i=1}^{l}\left(\omega''_\chi(\breve{h}_i^{\mathfrak{p}})\left(\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}\right)\right)\otimes\breve{\mu}(\bullet;\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}})
$$

that is a function in $\mathcal{S}(\breve{V}(\mathbb{A}_F)^2)$. Then

$$
(6.8)=\sum_{(\breve{x}_1,\breve{x}_2)\in\breve{V}^2}\left(\omega''_\chi\left(\imath(g_1^{\mathfrak{p}},g_2^{\mathfrak{p},\vee})\right)\Phi^{\mathrm{hor}}\right)((\breve{x}_1,\breve{x}_2)).
$$

We define the following theta series

$$
\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\bullet;\phi_1,\phi_2)=\log q\sum_{(\breve{x}_1,\breve{x}_2)\in\breve{V}^2}\left(\omega''_\chi(\bullet)\Phi^{\mathrm{hor}}\right)((\breve{x}_1,\breve{x}_2)),
$$

where q is the cardinality of the residue field of $E_{\mathfrak{p}^\circ}$. In summary, we have the following lemma.

Lemma 6.2.2. *Under the previous assumption, we have*

$$
\log q\left(\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{Z}_{\phi_2}(g_2)\right)=\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\imath(g_1,g_2^\vee);\phi_1,\phi_2).
$$

Second, we consider

$$
\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)=[\mathcal{Z}_{\phi_1}(g_1)]^\wedge_{\mathrm{sp}}\cdot\mathcal{V}_{\phi_2}(g_2)=[\mathcal{Z}_{\phi_1}(e_{\mathfrak{p}}g_1^{\mathfrak{p}})_{n,K^{\mathfrak{p}}}]^\wedge_{\mathrm{sp}}\cdot\mathcal{V}_{\phi_2}(g_2). \qquad (6.9)
$$

Let $\breve{h}_i^{\mathfrak{p}} \in \breve{H}_{\mathrm{fin}}^{\mathfrak{p}}$ $(i = 1, \ldots, l)$ and $h_{\mathfrak{p},j} \in K_{\mathfrak{p},0}$ $(j = 1, \ldots, m)$ be as above. Then

$$\begin{aligned}(6.9) &= \sum_{i=1}^{l}\sum_{j=1}^{m}[\mathcal{Z}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^{\mathfrak{p}})\phi_1}(e_{\mathfrak{p}}g_1^{\mathfrak{p}})_{n,K^{\mathfrak{p}}}]^{\wedge}_{s_n} \cdot \mathcal{V}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^{\mathfrak{p}})\phi_2}(g_2) \\ &= \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{\breve{x}_1\in\breve{V}}\sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^+\\(x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}}\left(\omega_\chi(g_1^{\mathfrak{p}})\phi_1^{\mathfrak{p}}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_1)\phi_{1,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_1)\mathscr{Z}(\breve{x}_1,x_1)_n\cdot\mathcal{V}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^{\mathfrak{p}})\phi_2}(g_2).\end{aligned} \tag{6.10}$$

The function

$$\nu(\breve{x}_1;\phi_{1,\mathfrak{p}},\phi_2,g_2) = \sum_{j=1}^{m}\sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^+\\(x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}}\phi_{1,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_1)\mathscr{Z}(\breve{x}_1,x_1)_n\cdot\mathcal{V}_{\omega_\chi(h_{\mathfrak{p},j})\phi_2}(g_2),$$

which is originally defined for $\breve{x}_1 \in \mathrm{Mor}\left((\mathbf{Y}, j_{\mathbf{Y}}), (\mathbf{X}, i_{\mathbf{X}})\right)_{\mathrm{reg}}$, can be extended by zero to a function in $\mathcal{S}(\breve{V}_{\mathfrak{p}})$. Then

$$(6.10) = \sum_{i=1}^{l}\sum_{\breve{x}_1\in\breve{V}}\left(\omega_\chi(g_1^{\mathfrak{p}})\phi_1^{\mathfrak{p}}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_1)\nu(\breve{x}_1;\phi_{1,\mathfrak{p}},\phi_2,g_2). \tag{6.11}$$

We define

$$\phi^{\mathrm{ver}} = \sum_{i=1}^{l}\left(\omega_\chi(\breve{h}_i^{\mathfrak{p}})\phi_1^{\mathfrak{p}}\right)\otimes\nu(\bullet;\phi_{1,\mathfrak{p}},\phi_2,g_2)$$

that is a function in $\mathcal{S}(\breve{V}(\mathbb{A}_F))$. Then

$$(6.11) = \sum_{\breve{x}_1\in\breve{V}}\left(\omega_\chi\left(g_1^{\mathfrak{p}}\right)\phi^{\mathrm{ver}}\right)(\breve{x}_1).$$

We define the following theta series

$$\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(\bullet;\phi_1,\phi_2,g_2) = \log q\sum_{\breve{x}_1\in\breve{V}}\left(\omega_\chi(\bullet)\phi^{\mathrm{ver}}\right)(\breve{x}_1).$$

In summary, we have the following lemma.

Lemma 6.2.3. *Under the previous assumptions, we have*

$$\log q\left(\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)\right) = \theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1;\phi_1,\phi_2,g_2),$$

which is a theta series for $g_1 \in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$.

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)=\log q\left(\mathcal{V}_{\phi_1}(g_1)\cdot\omega_K\right).$$

Then in summary, we have the following proposition.

Proposition 6.2.4. *For $\phi_{1,\mathsf{S}}\otimes\phi_{2,\mathsf{S}}\in\mathcal{S}(\mathbb{V}_{\mathsf{S}}^2)_{\mathrm{reg}}$ and $g_\alpha\in e_{\mathsf{S}}H'(\mathbb{A}_F^{\mathsf{S}})$ $(\alpha=1,2)$,*

$$\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}=\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\imath(g_1,g_2^\vee);\phi_1,\phi_2)+\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1,\phi_2,g_2)+A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)E(g_2,\phi_2).$$

6.3 Nonsplit case

In this section, we discuss the contribution of the local height pairing at a finite place $\mathfrak{p}$ in S that is nonsplit in E and such that $\epsilon(V_\mathfrak{p})=-1$. Let $\mathfrak{p}^\circ$ be the only place in $\Sigma^\circ_{\mathrm{fin}}$ lying over $\mathfrak{p}$. We identify $V_\mathfrak{p}$ with $B_\mathfrak{p}$, the unique division quaternion algebra over $F_\mathfrak{p}$, and the lattice Λ^- with $\mathcal{O}_{B_\mathfrak{p}}$.

6.3.1 Integral models and Čerednik–Drinfeld uniformization: minimal level

Let $K=K_\mathfrak{p}K^\mathfrak{p}$ be an open compact subgroup of $H(\mathbb{A}_{\mathrm{fin}})$ with $K^\mathfrak{p}$ sufficiently small and $K_\mathfrak{p}=K_{\mathfrak{p},n}$. In this subsection, we study the case where $n=0$. In 5.1.2, we introduce the integral model $\mathcal{M}_{0,K^\mathfrak{p}}$ defined over $\operatorname{Spec}\mathcal{O}_{E_{\mathfrak{p}^\circ}}$, whose generic fiber is $M_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$. We fix an $\mathbb{F}$-point s on the common neutral component of $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ and $\mathcal{M}_{0,K^\mathfrak{p}}$, which corresponds to a quintuple $(\mathbf{A},\theta_\mathbf{A},i_\mathbf{A},\bar\eta^p,\bar\eta^\mathfrak{p}_p)$. Repeating the process of 5.1.4, we obtain the E-hermitian space $\breve{V}$ with a fixed isometry $\gamma^\mathfrak{p}=(\gamma^\mathfrak{p}_p,\gamma^p):\breve{V}\otimes_F\mathbb{A}^\mathfrak{p}_{F,\mathrm{fin}}\to V\otimes_F\mathbb{A}^\mathfrak{p}_{F,\mathrm{fin}}$, and the reductive group $\breve{H}=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(\breve{V})$.

Let $\mathbf{X}=\mathcal{X}_s=(\mathbf{A}_{p^\infty})^2_1$, and $i_\mathbf{X}:\mathcal{O}_{B_\mathfrak{p}}\to\mathrm{End}(\mathbf{X})$ be the induced $\mathcal{O}_{B_\mathfrak{p}}$-action. Then $(\mathbf{X},i_\mathbf{X})$ is a *special formal* $\mathcal{O}_{B_\mathfrak{p}}$*-module* over $\operatorname{Spec}\mathbb{F}$ (*cf.* [Dri1976, BC1991, KR2000]). In particular $\mathbf{X}$ is of dimension 2 and height 4. Then $\mathrm{Aut}(\mathbf{X},i_\mathbf{X})$ acts on $\mathrm{Mor}\left((\mathbf{Y},j_\mathbf{Y}),(\mathbf{X},i_\mathbf{X})\right)$ and can be identified with $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$. For every integer h, we define a moduli functor $\underline{\Omega^h}$ on the category of schemes over $\operatorname{Spec}\mathcal{O}_{F^0_\mathfrak{p}}$ where ϖ is locally nilpotent: for every such scheme S, $\underline{\Omega^h}(S)$ is the set of equivalence classes of pairs (X,ρ_X) where

- X is a special formal $\mathcal{O}_{B_\mathfrak{p}}$-module over S;
- $\rho_X:\mathbf{X}\times_{\operatorname{Spec}\mathbb{F}}S_{\mathrm{sp}}\to X\times_{\operatorname{Spec}\mathbb{F}}S_{\mathrm{sp}}$ is a quasi-isogeny of height h of special formal $\mathcal{O}_{B_\mathfrak{p}}$-modules.

According to Drinfeld [Dri1976], this functor is represented by a formal scheme Ω^h over $\operatorname{Spf}\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}$. Let

$\Omega = \Omega^0$ [1] and $\Omega' = \Omega \times_{\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} \mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}}$.

If we denote by $[\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]^\wedge_{\mathrm{sp}}$ (resp. $[\mathcal{M}_{0,K^\mathfrak{p}}]^\wedge_{\mathrm{sp}}$) the formal completion along the special fiber, then we have the following Čerednik–Drinfeld uniformization (*cf.* [Dri1976, BC1991, RZ1996]):

$$[\mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]^\wedge_{\mathrm{sp}} \times_{\mathcal{O}_{F_\mathfrak{p}}} \mathcal{O}_{\widehat{F^0_\mathfrak{p}}} \cong \breve{H}^\dagger(\mathbb{Q})\backslash\left(\coprod_{h\in\mathbb{Z}} \Omega^h \times \prod_{i=2}^r B^\times_{\mathfrak{p}_i}/K^{\dagger,\mathfrak{p}}_p \times \breve{H}^\dagger(\mathbb{A}^p_{\mathrm{fin}})/K^{\dagger,p}\right);$$

$$[\mathcal{M}_{0,K^\mathfrak{p}}]^\wedge_{\mathrm{sp}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}} \cong \breve{H}(\mathbb{Q})\backslash\Omega' \times \breve{H}^\mathfrak{p}_{\mathrm{fin}}/K^\mathfrak{p}.$$

In fact, the above isomorphisms underly the corresponding uniformization for the universal p-divisible groups. For example, let $\mathcal{X}^{\mathrm{univ}} \to \Omega$ and $\mathcal{X}' \to \Omega'$ be the universal special formal $\mathcal{O}_{B_\mathfrak{p}}$-modules, respectively. Then we have the following commutative diagram:

$$\begin{tikzcd}
{[\mathcal{X}]^\wedge_{\mathrm{sp}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}}} \arrow[r, "\sim"] \arrow[d] & \breve{H}(\mathbb{Q})\backslash\mathcal{X}' \times \breve{H}^\mathfrak{p}_{\mathrm{fin}}/K^\mathfrak{p} \arrow[d] \\
{[\mathcal{M}_{0,K^\mathfrak{p}}]^\wedge_{\mathrm{sp}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}}} \arrow[r, "\sim"] & \breve{H}(\mathbb{Q})\backslash\Omega' \times \breve{H}^\mathfrak{p}_{\mathrm{fin}}/K^\mathfrak{p}.
\end{tikzcd}$$

For every nonzero element $\breve{x} \in \mathrm{Mor}((\mathbf{Y}, j_\mathbf{Y}), (\mathbf{X}, i_\mathbf{X}))$, we define a subfunctor $\underline{\mathscr{Z}(\breve{x})}$ of $\underline{\Omega^0}$ as follows: for every scheme S in the previously mentioned category, $\underline{\mathscr{Z}(\breve{x})}(S)$ is the set of equivalence classes of $(X, \rho_X) \in \underline{\Omega^0}(S)$ such that the following composed homomorphism

$$\left(\mathcal{Y} \times_{\mathrm{Spec}\,\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} \mathrm{Spec}\,\mathbb{F}\right) \times_{\mathrm{Spec}\,\mathbb{F}} S_{\mathrm{sp}} = \mathbf{Y} \times_{\mathrm{Spec}\,\mathbb{F}} S_{\mathrm{sp}} \xrightarrow{\breve{x}} \mathbf{X} \times_{\mathrm{Spec}\,\mathbb{F}} S_{\mathrm{sp}} \xrightarrow{\rho_X} X \times_S S_{\mathrm{sp}}$$

lifts to a homomorphism $\mathcal{Y} \times_{\mathrm{Spec}\,\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} S \to X$. Then $\underline{\mathscr{Z}(\breve{x})}$ is represented by a closed formal subscheme $\mathscr{Z}(\breve{x})$ of Ω^0. We denote by $\mathscr{Z}(\breve{x})^{\mathrm{hor}}$ the horizontal part of the associated divisor of $\mathscr{Z}(\breve{x})$, which is empty if $(\breve{x}, \breve{x}) = 0$.

6.3.2 Integral models and Čerednik–Drinfeld uniformization: higher level

Let $K = K_\mathfrak{p}K^\mathfrak{p}$ be an open compact subgroup of $H(\mathbb{A}_{\mathrm{fin}})$ with $K^\mathfrak{p}$ sufficiently small and $K_\mathfrak{p} = K_{\mathfrak{p},n}$ for n large such that $F^n_\mathfrak{p}$ contains $E_{\mathfrak{p}^\circ}$. Therefore, $M_K = M_{n,K^\mathfrak{p}}$.

Let $\Omega_{n,\eta} = \mathcal{X}^{\mathrm{univ}}_\eta[\varpi^n] - \mathcal{X}^{\mathrm{univ}}_\eta[\varpi^{n-1}]$ be the étale covering over the generic fiber Ω_η, viewed as a rigid space, with the Galois group $K^\dagger_{\mathfrak{p},0}/K^\dagger_{\mathfrak{p},n} = (\mathcal{O}_{B_\mathfrak{p}}/\varpi^n\mathcal{O}_{B_\mathfrak{p}})^\times$. Let Ω_n be the normalization of $\Omega \times_{\mathcal{O}_{\widehat{F^0_\mathfrak{p}}}} \mathcal{O}_{\widehat{F^n_\mathfrak{p}}}$ in $\Omega_{n,\eta} \times_{\widehat{F^0_\mathfrak{p}}} \widehat{F^n_\mathfrak{p}}$, whose set of connected components is parameterized by the coset

[1] This is not to be confused with the differential form Ω in Chapter 4.

$\mathcal{O}_{F_{\mathfrak{p}}}^{\times}/\nu(K_{\mathfrak{p},n}^{\dagger})$. Let Ω'_n be the neutral connected component of Ω_n, which is finite over $\Omega' \times_{\mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}$. The formal scheme Ω'_n is not regular but has double points. We replace Ω'_n by blowing up all its double points and denote by the same notation. It is easy to see that

$$\left(M_{n,K^{\mathfrak{p}}} \times_E \widehat{F^n_{\mathfrak{p}}}\right)^{\mathrm{rig}} \cong \breve{H}(\mathbb{Q})\backslash\left(\Omega'_{n,\eta} \times E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}\right).$$

The group $\breve{H}(\mathbb{Q})$ acts on Ω'_n by the universal property of normalization and blowing-up of double points. The quotient

$$\breve{H}(\mathbb{Q})\backslash\left(\Omega'_n \times E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}\right)$$

is regular, flat and projective over $\operatorname{Spf}\mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}$ (for sufficiently small $K^{\mathfrak{p}}$). By Grothendieck's Existence Theorem, we have a regular scheme $\mathcal{M}_{n,K^{\mathfrak{p}}}$ that is flat and projective over $\operatorname{Spec}\mathcal{O}_{F^n_{\mathfrak{p}}}$, and a morphism $p_n : \mathcal{M}_{n,K^{\mathfrak{p}}} \to \mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_{\mathfrak{p}}}$, such that the following diagram commutes:

$$\begin{tikzcd}
{[\mathcal{M}_{n,K^{\mathfrak{p}}}]^{\wedge}_{\mathrm{sp}}} \arrow[r, "\sim"] \arrow[d, "{[p_n]^{\wedge}_{\mathrm{sp}}}"'] & \breve{H}(\mathbb{Q})\backslash\left(\Omega'_n \times E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}\right) \arrow[d] \\
{[\mathcal{M}_{0,K^{\mathfrak{p}}}]^{\wedge}_{\mathrm{sp}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}} \arrow[r, "\sim"] & \left(\breve{H}(\mathbb{Q})\backslash\Omega' \times \breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}\right) \times_{\mathcal{O}_{\widehat{E^0_{\mathfrak{p}^\circ}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}.
\end{tikzcd}$$

Define $\mathscr{Z}'(\breve{x})_n$ by the following Cartesian diagram

$$\begin{tikzcd}
\mathscr{Z}'(\breve{x})_n \arrow[r] \arrow[d] & \Omega'_n \arrow[d] \\
\mathscr{Z}(\breve{x})^{\mathrm{hor}} \times_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}} \arrow[r] & \Omega \times_{\mathcal{O}_{\widehat{F^0_{\mathfrak{p}}}}} \mathcal{O}_{\widehat{F^n_{\mathfrak{p}}}}.
\end{tikzcd}$$

By the similar argument as in 6.2.1, we have the following decomposition

$$\mathscr{Z}'(\breve{x})_n = \bigcup_{\substack{x \in K^{\dagger}_{\mathfrak{p},n}\backslash\Lambda^- \\ \mathrm{Ht}(x)=\mathrm{Ht}(\breve{x})}} \mathscr{Z}(\breve{x}, x)_n$$

into (finitely many) formal divisors of Ω'_n.

In 5.1.5, we have defined a 1-dimensional closed subscheme $\mathcal{Z}(x_0)_{0,K^{\mathfrak{p}}}$ of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ whose generic fiber is $Z(x_0)_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, for a $K_{\mathfrak{p},0}K^{\mathfrak{p}}$-orbit $K_{\mathfrak{p},0}K^{\mathfrak{p}}x_0$ in $V(\mathbb{A}_{F,\mathrm{fin}})$ that is admissible. Define $\mathcal{Z}'(x_0)_{n,K^{\mathfrak{p}}}$ by the following Cartesian diagram

$$\begin{tikzcd}
\mathcal{Z}'(x_0)_{n,K^{\mathfrak{p}}} \arrow[r] \arrow[d] & \mathcal{M}_{n,K^{\mathfrak{p}}} \arrow[d] \\
\mathcal{Z}(x_0)^{\mathrm{hor}}_{0,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_{\mathfrak{p}}} \arrow[r] & \mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{F^n_{\mathfrak{p}}},
\end{tikzcd}$$

where $\mathcal{Z}(x_0)^{\mathrm{hor}}_{0,K^{\mathfrak{p}}}$ denotes the horizontal part of the associated divisor. For every orbit $K_{\mathfrak{p},n}K^{\mathfrak{p}}x$ inside $K_{\mathfrak{p},0}K^{\mathfrak{p}}x_0$, we define $\mathcal{Z}(x)_{n,K^{\mathfrak{p}}}$ to be the union of irreducible components of $\mathcal{Z}'(x_0)_{n,K^{\mathfrak{p}}}$ whose generic fiber contributes to the special divisor $Z(x)_{n,K^{\mathfrak{p}}} \times_{E_{\mathfrak{p}^\circ}} F^n_{\mathfrak{p}}$ on the generic fiber $M_{n,K^{\mathfrak{p}}} \times_E F^n_{\mathfrak{p}}$.

Let $\phi = \phi^0_\infty \otimes (\otimes_{v\in\Sigma_{\mathrm{fin}}}\phi_v)$ such that

- $\phi^{\mathfrak{p}}_{\mathrm{fin}} := \otimes_{v\in\Sigma_{\mathrm{fin}}-\{\mathfrak{p}\}}\phi_v$ is in $\mathcal{S}(V(\mathbb{A}^{\mathfrak{p}}_{F,\mathrm{fin}}))^{K^{\mathfrak{p}}}$;
- $\phi_{\mathfrak{p}}$ is in $\mathcal{S}(V_{\mathfrak{p}})_{\mathrm{reg}} \cap \mathcal{S}(V_{\mathfrak{p}})^{K^\dagger_{\mathfrak{p},n}}$, and its support is contained in Λ^-.

In particular, $\phi(0) = 0$. Consider the generating series

$$Z_\phi(e_{\mathfrak{p}}g^{\mathfrak{p}}) = \sum_{x\in K_{\mathfrak{p},n}K^{\mathfrak{p}}\backslash V(\mathbb{A}_{F,\mathrm{fin}})-\{0\}} \left(\phi_{\mathfrak{p}} \otimes (\omega_\chi(g^{\mathfrak{p}})\phi^{\mathfrak{p}})\right)(x) Z(x)_{K_{\mathfrak{p},n}K^{\mathfrak{p}}}.$$

Define

$$\mathcal{Z}_\phi(e_{\mathfrak{p}}g^{\mathfrak{p}})_{n,K^{\mathfrak{p}}} = \sum_{x\in K_{\mathfrak{p},n}K^{\mathfrak{p}}\backslash V(\mathbb{A}_{F,\mathrm{fin}})-\{0\}} \left(\phi_{\mathfrak{p}} \otimes (\omega_\chi(g^{\mathfrak{p}})\phi^{\mathfrak{p}})\right)(x) \mathcal{Z}(x)_{n,K^{\mathfrak{p}}},$$

whose generic fiber is the base change of $Z_\phi(e_{\mathfrak{p}}g^{\mathfrak{p}})$ to $F^n_{\mathfrak{p}}$. Moreover, its completion along the neutral component is (the image of)

$$\sum_{\breve{x}\in\breve{V}} \sum_{\substack{x\in K_{\mathfrak{p},n}\backslash\Lambda^+ \\ (x,x)=(\breve{x},\breve{x})'}} \left(\omega_\chi(g^{\mathfrak{p}})\phi^{\mathfrak{p}}\right)(\breve{x})\phi_{\mathfrak{p}}(x)\mathscr{Z}(\breve{x},x)_n. \tag{6.12}$$

6.3.3 Coherence for intersection numbers

Let $\mathrm{Mor}\left((\mathbf{Y},j_{\mathbf{Y}}),(\mathbf{X},i_{\mathbf{X}})\right)_{\mathrm{reg}} \subset \mathrm{Mor}\left((\mathbf{Y},j_{\mathbf{Y}}),(\mathbf{X},i_{\mathbf{X}})\right)$ be the subset consisting of $\breve{x}$ with $(\breve{x},\breve{x}) \neq 0$. Define the subset

$$\mathrm{Red}^{\mathrm{ns}}_n = \{(x_1,x_2;\breve{x}_1,\breve{x}_2)\} \subset \left(\Lambda^-\right)^2 \times \mathrm{Mor}\left((\mathbf{Y},j_{\mathbf{Y}}),(\mathbf{X},i_{\mathbf{X}})\right)^2_{\mathrm{reg}}$$

by the conditions that

- $\mathrm{Ht}(x_\alpha) = \mathrm{Ht}(\breve{x}_\alpha)$ for $\alpha = 1, 2$;
- $K^\dagger_{\mathfrak{p},n} x_1$ and $K^\dagger_{\mathfrak{p},n} x_2$ are linearly independent.

Then by definition, for $(x_1, x_2; \breve{x}_1, \breve{x}_2) \in \mathrm{Red}^{\mathrm{ns}}_n$, the formal divisors $\mathscr{Z}(\breve{x}_1, x_1)_n$ and $\mathscr{Z}(\breve{x}_2, x_2)_n$ will intersect properly. We let

$$m(x_1, x_2; \breve{x}_1, \breve{x}_2) = \mathscr{Z}(\breve{x}_1, x_1)_n \cdot \mathscr{Z}(\breve{x}_2, x_2)_n$$

be the intersection multiplicity, which is a well-defined continuous function on $\mathrm{Red}^{\mathrm{ns}}_n$. The following lemma is straightforward.

Lemma 6.3.1. *Suppose that for $\alpha = 1, 2$, $\phi_{\alpha,\mathfrak{p}} \in \mathcal{S}(V_\mathfrak{p})_{\mathrm{reg}} \cap \mathcal{S}(V_\mathfrak{p})^{K^\dagger_{\mathfrak{p},n}}$ whose support is contained in Λ^- and such that $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}$ is in $\mathcal{S}(V^2_\mathfrak{p})_{\mathrm{reg}}$. Then the following function*

$$\mu(\breve{x}_1, \breve{x}_2; \phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}) = \sum_{\substack{x_1 \in K_{\mathfrak{p},n} \backslash \Lambda^- \\ (x_1, x_1) = (\breve{x}_1, \breve{x}_1)'}} \sum_{\substack{x_2 \in K_{\mathfrak{p},n} \backslash \Lambda^- \\ (x_2, x_2) = (\breve{x}_2, \breve{x}_2)'}} (\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}})(x_1, x_2) m(x_1, x_2; \breve{x}_1, \breve{x}_2),$$

which is a priori *defined on* $\mathrm{Mor}\left((\mathbf{Y}, j_\mathbf{Y}), (\mathbf{X}, i_\mathbf{X})\right)^2_{\mathrm{reg}}$, *is a Schwartz function in $\mathcal{S}(\breve{V}^2_\mathfrak{p})$ via extension by zero.*

The remaining discussion follows similarly as in 6.2.2. We sketch the process. Let $h_{\mathfrak{p},j} \in K_{\mathfrak{p},0}$ $(j = 1, \dots, m)$ be similarly defined previously. Let

$$\breve{\mu}(\breve{x}_1, \breve{x}_2; \phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}}) = \sum_{j=1}^m \mu\left(\breve{x}_1, \breve{x}_2; \omega''_\chi(h_{\mathfrak{p},j})(\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}})\right),$$

whose value does not depend on the representatives we choose.

We return to our original assumption for this chapter that $\phi_{\alpha,\mathsf{S}} \in \mathcal{S}(\mathbb{V}_\mathsf{S})_{\mathrm{reg}}$ $(\alpha = 1, 2)$, $\phi_{1,\mathsf{S}} \otimes \phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}^2_\mathsf{S})_{\mathrm{reg}}$, and $g_\alpha \in e_\mathsf{S} H'(\mathbb{A}^\mathsf{S}_F)$ for $\alpha = 1, 2$. Choose some element $\mathbf{e}_\alpha \in E^\times$ such that $\omega_\chi(m(\mathbf{e}_\alpha))\phi_{\alpha,\mathfrak{p}}$ is supported on Λ^+ for $\alpha = 1, 2$. We have

$$Z_{\phi_\alpha}(g_\alpha) = \sum_{x_\alpha \in K \backslash \mathbb{V}_{\mathrm{fin}}} (\omega_\chi(g_\alpha)\phi_\alpha)(x_\alpha \mathbf{e}_\alpha) Z(x_\alpha)_K.$$

Therefore, we can add one more assumption that $\phi_{\alpha,\mathfrak{p}}$ is in $\mathcal{S}(V_\mathfrak{p})_{\mathrm{reg}} \cap \mathcal{S}(V_\mathfrak{p})^{K^\dagger_{\mathfrak{p},n}}$, and its support is contained in Λ^-. We let $\mathcal{Z}_{\phi_\alpha}(g_\alpha) = \mathcal{Z}_\phi(e_\mathfrak{p} g^\mathfrak{p}_\alpha)_{n,K^\mathfrak{p}}$ and denote by $[\mathcal{Z}_\phi(g_\alpha)]^\wedge_{\mathrm{sp}}$ its completion along the special fiber. We have a similar decomposition as (6.2).

First, we consider

$$\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{Z}_{\phi_2}(g_2)=[\mathcal{Z}_{\phi_1}(g_1)]^{\wedge}_{\mathrm{sp}}\cdot[\mathcal{Z}_{\phi_2}(g_2)]^{\wedge}_{\mathrm{sp}}=[\mathcal{Z}_{\phi_1}(e_{\mathfrak{p}}g_1^{\mathfrak{p}})_{n,K^{\mathfrak{p}}}]^{\wedge}_{\mathrm{sp}}\cdot[\mathcal{Z}_{\phi_2}(e_{\mathfrak{p}}g_2^{\mathfrak{p}})_{n,K^{\mathfrak{p}}}]^{\wedge}_{\mathrm{sp}}. \tag{6.13}$$

Let $\breve{h}_i^{\mathfrak{p}}\in\breve{H}^{\mathfrak{p}}_{\mathrm{fin}}$ $(i=1,\dots,l)$ be a set of representatives of the double coset $\breve{H}(\mathbb{Q})\backslash\breve{H}^{\mathfrak{p}}_{\mathrm{fin}}/K^{\mathfrak{p}}$. We assume that $\breve{h}_1^{\mathfrak{p}}$ is the identity.

Then

$$(6.13)=\sum_{i=1}^{l}\sum_{j=1}^{m}[\mathcal{Z}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^{\mathfrak{p}})\phi_1}(e_{\mathfrak{p}}g_1^{\mathfrak{p}})_{n,K^{\mathfrak{p}}}]^{\wedge}_{h_1}\cdot[\mathcal{Z}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^{\mathfrak{p}})\phi_2}(e_{\mathfrak{p}}g_2^{\mathfrak{p}})_{n,K^{\mathfrak{p}}}]^{\wedge}_{h_1}, \tag{6.14}$$

where the subscript h_1 means that we take completion along the component indexed by h_1. By (6.12),

$$\begin{aligned}(6.14)=&\sum_{i=1}^{l}\sum_{j=1}^{m}\left(\sum_{\breve{x}_1\in\breve{V}}\sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^-\\(x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}}\left(\omega_\chi(g_1^{\mathfrak{p}})\phi_1^{\mathfrak{p}}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_1)\phi_{1,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_1)\mathscr{Z}(\breve{x}_1,x_1)_n\right)\\&\cdot\left(\sum_{\breve{x}_2\in\breve{V}}\sum_{\substack{x_2\in K_{\mathfrak{p},n}\backslash\Lambda^-\\(x_2,x_2)=(\breve{x}_2,\breve{x}_2)'}}\left(\omega_\chi(g_2^{\mathfrak{p}})\phi_2^{\mathfrak{p}}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_2)\phi_{2,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_2)\mathscr{Z}(\breve{x}_2,x_2)_n\right)\\=&\sum_{i=1}^{l}\sum_{(\breve{x}_1,\breve{x}_2)\in\breve{V}^2}\left(\omega''_\chi\left(\imath(g_1^{\mathfrak{p}},g_2^{\mathfrak{p},\vee})\right)\left(\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}\right)\right)(\breve{h}_i^{\mathfrak{p},-1}(\breve{x}_1,\breve{x}_2))\breve{\mu}(\breve{x}_1,\breve{x}_2;\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}).\end{aligned} \tag{6.15}$$

We define

$$\Phi^{\mathrm{hor}}=\sum_{i=1}^{l}\left(\omega''_\chi(\breve{h}_i^{\mathfrak{p}})\left(\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}\right)\right)\otimes\breve{\mu}(\bullet;\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}})$$

that is a function in $\mathcal{S}(\breve{V}(\mathbb{A}_F)^2)$. Then

$$(6.15)=\sum_{(\breve{x}_1,\breve{x}_2)\in\breve{V}^2}\left(\omega''_\chi\left(\imath(g_1^{\mathfrak{p}},g_2^{\mathfrak{p},\vee})\right)\Phi^{\mathrm{hor}}\right)((\breve{x}_1,\breve{x}_2)).$$

We define the following theta series

$$\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\bullet;\phi_1,\phi_2)=\log q\sum_{(\breve{x}_1,\breve{x}_2)\in\breve{V}^2}\left(\omega''_\chi(\bullet)\Phi^{\mathrm{hor}}\right)((\breve{x}_1,\breve{x}_2)),$$

where q is the cardinality of the residue field of $E_{\mathfrak{p}^\circ}$. In summary, we have the following lemma.

Lemma 6.3.2. *Under the previous assumption, we have*

$$\log q\left(\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{Z}_{\phi_2}(g_2)\right)=\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\imath(g_1,g_2^\vee);\phi_1,\phi_2).$$

Second, we consider

$$\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)=[\mathcal{Z}_{\phi_1}(g_1)]^\wedge_{\mathrm{sp}}\cdot\mathcal{V}_{\phi_2}(g_2)=[\mathcal{Z}_{\phi_1}(e_\mathfrak{p}g_1^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{\mathrm{sp}}\cdot\mathcal{V}_{\phi_2}(g_2). \tag{6.16}$$

Let $\breve{h}_i^\mathfrak{p}\in\breve{H}^\mathfrak{p}_{\mathrm{fin}}$ $(i=1,\dots,l)$ and $h_{\mathfrak{p},j}\in K_{\mathfrak{p},0}$ $(j=1,\dots,m)$ be as above. Then

$$\begin{aligned}(6.16)&=\sum_{i=1}^{l}\sum_{j=1}^{m}[\mathcal{Z}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^\mathfrak{p})\phi_1}(e_\mathfrak{p}g_1^\mathfrak{p})_{n,K^\mathfrak{p}}]^\wedge_{h_1}\cdot\mathcal{V}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^\mathfrak{p})\phi_2}(g_2)\\&=\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{\breve{x}_1\in\breve{V}}\sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^-\\(x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}}\left(\omega_\chi(g_1^\mathfrak{p})\phi_1^\mathfrak{p}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_1)\phi_{1,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_1)\mathscr{X}(\breve{x}_1,x_1)_n\cdot\mathcal{V}_{\omega_\chi(h_{\mathfrak{p},j}\breve{h}_i^\mathfrak{p})\phi_2}(g_2).\end{aligned}\tag{6.17}$$

The function

$$\nu(\breve{x}_1;\phi_{1,\mathfrak{p}},\phi_2,g_2)=\sum_{j=1}^{m}\sum_{\substack{x_1\in K_{\mathfrak{p},n}\backslash\Lambda^-\\(x_1,x_1)=(\breve{x}_1,\breve{x}_1)'}}\phi_{1,\mathfrak{p}}(h_{\mathfrak{p},j}^{-1}x_1)\mathscr{X}(\breve{x}_1,x_1)_n\cdot\mathcal{V}_{\omega_\chi(h_{\mathfrak{p},j})\phi_2}(g_2),$$

which is originally defined for $\breve{x}_1\in\mathrm{Mor}\left((\mathbf{Y},j_\mathbf{Y}),(\mathbf{X},i_\mathbf{X})\right)_{\mathrm{reg}}$, can be extended by zero to a function in $\mathcal{S}(\breve{V}_\mathfrak{p})$. Then

$$(6.17)=\sum_{i=1}^{l}\sum_{\breve{x}_1\in\breve{V}}\left(\omega_\chi(g_1^\mathfrak{p})\phi_1^\mathfrak{p}\right)(\breve{h}_i^{\mathfrak{p},-1}\breve{x}_1)\nu(\breve{x}_1;\phi_{1,\mathfrak{p}},\phi_2,g_2). \tag{6.18}$$

We define

$$\phi^{\mathrm{ver}}=\sum_{i=1}^{l}\left(\omega_\chi(\breve{h}_i^\mathfrak{p})\phi_1^\mathfrak{p}\right)\otimes\nu(\bullet;\phi_{1,\mathfrak{p}},\phi_2,g_2)$$

that is a function in $\mathcal{S}(\breve{V}(\mathbb{A}_F))$. Then

$$(6.18)=\sum_{\breve{x}_1\in\breve{V}}\left(\omega_\chi\left(g_1^\mathfrak{p}\right)\phi^{\mathrm{ver}}\right)(\breve{x}_1).$$

We define the following theta series

$$\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(\bullet;\phi_1,\phi_2,g_2)=\log q\sum_{\breve{x}_1\in\breve{V}}\left(\omega_\chi(\bullet)\phi^{\mathrm{ver}}\right)(\breve{x}_1).$$

In summary, we have the following lemma.

Lemma 6.3.3. *Under the previous assumption, we have*

$$\log q\left(\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)\right)=\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1;\phi_1,\phi_2,g_2)$$

that is a theta series for $g_1\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$.

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)=\log q\left(\mathcal{V}_{\phi_1}(g_1)\cdot\omega_K\right).$$

Then in summary, we have the following proposition.

Proposition 6.3.4. *For* $\phi_{1,\mathsf{S}}\otimes\phi_{2,\mathsf{S}}\in\mathcal{S}(\mathbb{V}^2_{\mathsf{S}})_{\mathrm{reg}}$ *and* $g_\alpha\in e_{\mathsf{S}}H'(\mathbb{A}_F^{\mathsf{S}})$ *(*$\alpha=1,2$*),*

$$\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}=\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\imath(g_1,g_2^\vee);\phi_1,\phi_2)+\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1,\phi_2,g_2)+A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)E(g_2,\phi_2).$$

Chapter 7

Arithmetic inner product formula: the main theorem

We will accomplish the proof of the main theorem, *i.e.* the arithmetic inner product formula for $n = 1$ in this chapter. In 7.1, we introduce the general theory of holomorphic projections for the group $\mathrm{U}(1,1)_F$, and compute such projection for the analytic kernel function. In particular, this process will relate the archimedean local terms to admissible height pairings (at archimedean places) and leave local terms at finite places unchanged. Based on all these previous results, we come to the final stage of the proof in 7.2.

7.1 Holomorphic projection

In this section, we define and compute the holomorphic projection of the analytic kernel function $E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$, and study its relation with the geometric kernel function when $n = 1$. We follow the general theory in the case of GL_2 in [GZ1986, Zha2001a, Zha2001b, YZZa]. In what follows, $n = 1$ and $H' = H_1$ in particular.

We remind the reader that we fix an additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ such that for every $\iota \in \Sigma_\infty$, $\psi_\iota(t) = \exp(2\pi i t)$ ($t \in F_\iota = \mathbb{R}$) at the beginning of Chapter 3.

7.1.1 Holomorphic and quasi-holomorphic projection: generality

Let $\mathfrak{k} = (\mathfrak{k}_\iota)_\iota \in \mathbb{Z}^{\Sigma_\infty}$ be a sequence of integers. We denote by $\mathcal{A}_0(H') \subset \mathcal{A}(H')$ the subspace of cusp forms. Let $\mathcal{A}_0^{\mathfrak{k}}(H') \subset \mathcal{A}_0(H')$ be the subspace consisting of those cusp forms of weight $(1 + \mathfrak{k}, 1 - \mathfrak{k})$ (*cf.* Definition 2.1.2). Let Z' be the center of H', which is isomorphic to $E^{\times,1}$, as an F-torus. We define a character $\zeta^{\mathfrak{k}}$ of Z'_∞ by letting $\zeta_\iota^{\mathfrak{k}}(z_\iota) = z_\iota^{2\mathfrak{k}_\iota}$. Let $\mathcal{A}(H', \zeta^{\mathfrak{k}}) \subset \mathcal{A}(H')$ be the subspace consisting of the forms that have archimedean central character $\zeta^{\mathfrak{k}}$. It is clear that $\mathcal{A}_0^{\mathfrak{k}}(H') \subset \mathcal{A}(H', \zeta^{\mathfrak{k}})$. Recall that $F^+ \subset F$ is the subset of totally positive elements. For each fixed $t \in F^+$, the t-th archimedean Whittaker coefficients (with respect to the ψ_∞) of all members of $\mathcal{A}_0^{\mathfrak{k}}(H')$ are the same one, that is, the function $W_t^{\mathfrak{k}}$ defined by

$$W_t^{\mathfrak{k}}\left(n(b)m(a)[k_1, k_2]\right) = \prod_{\iota \in \Sigma_\infty} \exp\left(2\pi i t\left(b_\iota + i a_\iota \overline{a_\iota}\right)\right)\left(a_\iota \overline{a_\iota}\right) k_{1,\iota}^{1+\mathfrak{k}_\iota} k_{2,\iota}^{1-\mathfrak{k}_\iota},$$

where $a = (a_\iota) \in E_\infty^\times$, $b = (b_\iota) \in F_\infty$, and $[k_1, k_2] = ([k_{1,\iota}, k_{2,\iota}]) \in \mathcal{K}'_\infty$.

We denote by $\mathcal{A}_0^{\mathfrak{k}}(H' \times H') \subset \mathcal{A}_0(H' \times H')$ the subspace consisting of cusp forms $\mathbf{f}(\bullet, \bullet)$ such that for every $g \in H'(\mathbb{A}_F)$, both $\mathbf{f}(g, \bullet)$ and $\mathbf{f}(\bullet, g)$ are in $\mathcal{A}_0^{\mathfrak{k}}(H')$. The space $\mathcal{A}_0(H' \times H')$ is a Hilbert space with norm given by the Petersson inner product $\langle -, - \rangle_{H' \times H'}$.

Definition 7.1.1 (Holomorphic projection)**.** We define a linear map

$$\mathrm{Pr} : \mathcal{A}(H' \times H', \zeta^{\mathfrak{k}}) \to \mathcal{A}_0^{\mathfrak{k}}(H' \times H'),$$

which sends $\mathbf{f}$ to the unique element $\mathrm{Pr}(\mathbf{f})$ in the latter space satisfying $\langle \mathbf{f}, \mathbf{f}' \rangle_{H' \times H'} = \langle \mathrm{Pr}(\mathbf{f}), \mathbf{f}' \rangle_{H' \times H'}$ for every $\mathbf{f}' \in \mathcal{A}_0^{\mathfrak{k}}(H' \times H')$.

Let ψ' be a character of $F\backslash\mathbb{A}_F$ such that $\psi'_\infty(x) = \psi_\infty(tx)$ for some $t \in F_\infty^\times$. For an automorphic form $\mathbf{f} \in \mathcal{A}(H' \times H', \zeta^{\mathfrak{k}})$, we define

$$\mathbf{f}_{\psi',s}(g_1, g_2) = (4\pi)^{2d} W_t^{\mathfrak{k}}(g_{1,\infty}) W_t^{\mathfrak{k}}(g_{2,\infty}) \iint_{[Z'(F_\infty)N'(F_\infty)\backslash H'(F_\infty)]^2} \lambda_{P'}(h_1)^s \lambda_{P'}(h_2)^s$$
$$\mathbf{f}_{\psi'}(h_1 g_{1,\mathrm{fin}}, h_2 g_{2,\mathrm{fin}}) \overline{W_t^{\mathfrak{k}}(h_1) W_t^{\mathfrak{k}}(h_2)} \mathrm{d}h_1 \mathrm{d}h_2,$$

where $\mathbf{f}_{\psi'}$ is the ψ'-Whittaker coefficient of $\mathbf{f}$.

Proposition 7.1.2. *Suppose that* $\mathbf{f} \in \mathcal{A}(H' \times H', \zeta^{\mathfrak{k}})$ *has the following asymptotic behavior: for some* $\epsilon > 0$,

$$\mathbf{f}(m(a_1)g_1, m(a_2)g_2) = O_{g_1,g_2}\left(|a_1 a_2|_{\mathbb{A}_E}^{1-\epsilon}\right)$$

as $|a_1 a_2|_{\mathbb{A}_E} \to \infty$ *for* $a_\alpha \in \mathbb{A}_E^\times$ $(\alpha = 1, 2)$*. Then the holomorphic projection* $\mathrm{Pr}(\mathbf{f})$ *has the* ψ'*-Whittaker coefficient*

$$\mathrm{Pr}(\mathbf{f})_{\psi'}(g_1, g_2) = \lim_{s \to 0} \mathbf{f}_{\psi',s}(g_1, g_2).$$

Proof. Let ζ_α $(\alpha = 1, 2)$ be two automorphic characters of $Z'(\mathbb{A}_F)$ such that $\zeta_{\alpha,\infty} = \zeta^{\mathfrak{k}}$. Consider two ψ'-Whittaker functions $W^\alpha(g_\alpha) = W_t^{\mathfrak{k}}(g_{\alpha,\infty}) W_{\mathrm{fin}}^\alpha(g_{\alpha,\mathrm{fin}})$ $(\alpha = 1, 2)$ of $H'(\mathbb{A}_F)$ with central character ζ_α, such that W_{fin}^α is compactly supported modulo $Z'(\mathbb{A}_{F,\mathrm{fin}})N'(\mathbb{A}_{F,\mathrm{fin}})$. We define the Poincaré series to be

$$P_{W^\alpha}(g_\alpha) = \lim_{s \to 0^+} \sum_{\gamma \in Z'(F)N'(F)\backslash H'(F)} W^\alpha(\gamma g_\alpha) \lambda_{P'}(\gamma_\infty g_{\alpha,\infty})^s.$$

For a function $\mathbf{f}' \in \mathcal{A}(H' \times H', \zeta^{\mathfrak{k}})$, we let

$$\mathbf{f}'_{\zeta_1,\zeta_2}(g_1, g_2) = \iint_{[Z'(F)\backslash Z'(\mathbb{A}_F)]^2} \mathbf{f}'(g_1 z_1, g_2 z_2) \zeta_1^{-1}(z_1) \zeta_2^{-1}(z_2) \mathrm{d}z_1 \mathrm{d}z_2.$$

Assume that $\mathbf{f}$ has the asymptotic behavior as in the proposition. Then

$$\begin{aligned}
&\langle \mathbf{f}, P_{W^1} \otimes P_{W^2} \rangle_{H' \times H'} \\
= &\iint_{[Z'(\mathbb{A}_F)H'(F)\backslash H'(\mathbb{A}_F)]^2} \mathbf{f}_{\zeta_1,\zeta_2}(g_1, g_2) \overline{P_{W^1}(g_1) P_{W^2}(g_2)} \mathrm{d}g_1 \mathrm{d}g_2 \\
= &\lim_{s \to 0^+} \iint_{[Z'(\mathbb{A}_F)N'(F)\backslash H'(\mathbb{A}_F)]^2} \mathbf{f}_{\zeta_1,\zeta_2}(g_1, g_2) \overline{W^1(g_1) W^2(g_2)} \lambda_{P'}(g_{1,\infty})^s \lambda_{P'}(g_{2,\infty})^s \mathrm{d}g_1 \mathrm{d}g_2 \\
= &\lim_{s \to 0^+} \iint_{[Z'(\mathbb{A}_F)N'(\mathbb{A}_F)\backslash H'(\mathbb{A}_F)]^2} (\mathbf{f}_{\zeta_1,\zeta_2})_\psi(g_1, g_2) \overline{W^1(g_1) W^2(g_2)} \lambda_{P'}(g_{1,\infty})^s \lambda_{P'}(g_{2,\infty})^s \mathrm{d}g_1 \mathrm{d}g_2. \quad (7.1)
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\langle \mathrm{Pr}(\mathbf{f}), P_{W^1} \otimes P_{W^2} \rangle_{H' \times H'} \\
=& \iint_{[Z'(F_\infty)N'(F_\infty)\backslash H'(F_\infty)]^2} W_t^{\mathfrak{k}}(g_1) W_t^{\mathfrak{k}}(g_2) \overline{W_t^{\mathfrak{k}}(g_1) W_t^{\mathfrak{k}}(g_2)} \mathrm{d}g_1 \mathrm{d}g_2 \\
& \iint_{[Z'(\mathbb{A}_{F,\mathrm{fin}})N'(\mathbb{A}_{F,\mathrm{fin}})\backslash H'(\mathbb{A}_{F,\mathrm{fin}})]^2} (\mathrm{Pr}(\mathbf{f})_{\zeta_1,\zeta_2})_{\psi'}(g_{1,\mathrm{fin}}, g_{2,\mathrm{fin}}) \overline{W^1_{\mathrm{fin}}(g_{1,\mathrm{fin}}) W^2_{\mathrm{fin}}(g_{2,\mathrm{fin}})} \mathrm{d}g_{1,\mathrm{fin}} \mathrm{d}g_{2,\mathrm{fin}} \\
=& (4\pi)^{-2d} \iint_{[Z'(\mathbb{A}_{F,\mathrm{fin}})N'(\mathbb{A}_{F,\mathrm{fin}})\backslash H'(\mathbb{A}_{F,\mathrm{fin}})]^2} (\mathrm{Pr}(\mathbf{f})_{\zeta_1,\zeta_2})_{\psi'}(g_{1,\mathrm{fin}}, g_{2,\mathrm{fin}}) \overline{W^1_{\mathrm{fin}}(g_{1,\mathrm{fin}}) W^2_{\mathrm{fin}}(g_{2,\mathrm{fin}})} \mathrm{d}g_{1,\mathrm{fin}} \mathrm{d}g_{2,\mathrm{fin}}.
\end{aligned}
\tag{7.2}
$$

Since, by definition, $\langle \mathrm{Pr}(\mathbf{f}), P_{W^1} \otimes P_{W^2} \rangle_{H' \times H'} = \langle \mathbf{f}, P_{W^1} \otimes P_{W^2} \rangle_{H' \times H'}$, (7.1) and (7.2) are equal for all (ζ_1, ζ_2) and $(W^1_{\mathrm{fin}}, W^2_{\mathrm{fin}})$. Therefore, the proposition follows. □

For general $\mathbf{f} \in \mathcal{A}(H' \times H', \zeta^{\mathfrak{k}})$ that may not have the asymptotic behavior as in Proposition 7.1.2, we propose the following definition.

Definition 7.1.3. For $\mathbf{f} \in \mathcal{A}(H' \times H', \zeta^{\mathfrak{k}})$, we define

$$\widetilde{\mathrm{Pr}}(\mathbf{f})_{\psi'}(g_1, g_2) = \mathrm{const}_{s=0}\, \mathbf{f}_{\psi',s}(g_1, g_2),$$

where $\mathrm{const}_{s=a}$ denotes the constant term at $s = a$ (possibly after meromorphic continuation around a). We define the *quasi-holomorphic projection* of $\mathbf{f}$ to be

$$\widetilde{\mathrm{Pr}}(\mathbf{f})(g_1, g_2) = \sum_{\psi'} \widetilde{\mathrm{Pr}}(\mathbf{f})_{\psi'}(g_1, g_2),$$

where the sum is taken over all nontrivial characters of $F\backslash\mathbb{A}_F$.

In fact, the same definition still makes sense if $\mathbf{f}$ is only left invariant under $N'(F) \times N'(F)$. In some middle steps of the following calculation, we need to apply $\widetilde{\mathrm{Pr}}$ to such functions or even currents.

Then Proposition 7.1.2 amounts equivalently to say that if $\mathbf{f}$ satisfies the asymptotic behavior there, we have $\widetilde{\mathrm{Pr}}(\mathbf{f}) = \mathrm{Pr}(\mathbf{f})$.

7.1.2 Siegel–Fourier expansion of Eisenstein series on $\mathrm{U}(2,2)$

We study the (Siegel–)Fourier expansion of the Eisenstein series $E(s,g,\Phi)$ at $g=e\in H''(\mathbb{A}_F)=H_2(\mathbb{A}_F)$ for $\Phi=\phi_1\otimes\phi_2\in\mathcal{S}(\mathbb{V}^2)$, where $\mathbb{V}$ is a hermitian space of rank 2 over $\mathbb{A}_E$. By definition

$$E(s,g,\phi_1\otimes\phi_2)=\sum_{\gamma\in P(F)\backslash H''(F)}\left(\omega_\chi(\gamma g)(\phi_1\otimes\phi_2)\right)(0)\lambda_P(\gamma g)^s,$$

which is absolutely convergent when $\operatorname{Re}s>1$. The following lemma is straightforward.

Lemma 7.1.4. *The group $H''(F)$ is the disjoint union of the following cosets:*

1.

$$P(F)\mathsf{w}_2\begin{pmatrix}1 & & b_{11} & b_{12}\\ & 1 & b_{21} & b_{22}\\ & & 1 & \\ & & & 1\end{pmatrix},\qquad b=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\in\mathrm{Her}_2(E);$$

2.

$$P(F)\mathsf{w}_{2,1}\begin{pmatrix}1 & & 0 & 0\\ c & 1 & 0 & b_{22}\\ & & 1 & -c^\tau\\ & & & 1\end{pmatrix},\qquad c\in E,\ b_{22}\in F;$$

3. $P(F)$,

where we recall that

$$\mathsf{w}_2=\begin{pmatrix} & \mathbf{1}_2\\ -\mathbf{1}_2 & \end{pmatrix};\qquad \mathsf{w}_{2,1}=\begin{pmatrix}1 & & & \\ & & & 1\\ & & 1 & \\ & -1 & & \end{pmatrix}.$$

In particular, we have

$$E(s,g,\phi_1\otimes\phi_2)=E^2(s,g,\phi_1\otimes\phi_2)+E^0(s,g,\phi_1\otimes\phi_2)+\sum_{c\in E}E^{1,c}(s,g,\phi_1\otimes\phi_2),$$

where

$$E^2(s,g,\phi_1\otimes\phi_2)=\sum_{b\in\mathrm{Her}_2(E)}\left(\omega_\chi(\mathsf{w}_2 n(b)g)(\phi_1\otimes\phi_2)\right)(0)\lambda_P(\mathsf{w}_2 n(b)g)^s;$$

$$E^{1,c}(s,g,\phi_1\otimes\phi_2)=\sum_{b_{22}\in F}\left(\omega_\chi\left(\mathsf{w}_{2,1}m\left(\begin{pmatrix}1&\\c&1\end{pmatrix}\right)n\left(\begin{pmatrix}0&0\\0&b_{22}\end{pmatrix}\right)g\right)(\phi_1\otimes\phi_2)\right)(0)$$
$$\lambda_P\left(\mathsf{w}_{2,1}m\left(\begin{pmatrix}1&\\c&1\end{pmatrix}\right)n\left(\begin{pmatrix}0&0\\0&b_{22}\end{pmatrix}\right)g\right)^s$$

$$E^0(s,g,\phi_1\otimes\phi_2)=\left(\omega(g)(\phi_1\otimes\phi_2)\right)(0)\lambda_P(g)^s.$$

They are all invariant under the left translation by $N(F)$.

Let T be an element in $\mathrm{Her}_2(E)$. We calculate

$$E_T^\beta(s,e,\phi_1\otimes\phi_2):=\int_{N(F)\backslash N(\mathbb{A}_F)}E^\beta(s,n(b'),\phi_1\otimes\phi_2)\psi(\mathrm{tr}\,Tb')^{-1}\mathrm{d}n$$

for $\beta=2,0$, and $1,c$ with $c\in E$, respectively.

For $\beta=2$, we have

$$E_T^2(s,e,\phi_1\otimes\phi_2)=\int_{N(\mathbb{A}_F)}\left(\omega_\chi(\mathsf{w}_2 n(b'))(\phi_1\otimes\phi_2)\right)(0)\lambda_P(\mathsf{w}_2 n(b'))^s\psi(\mathrm{tr}\,Tb')^{-1}\mathrm{d}n. \tag{7.3}$$

For $\beta=0$, we have

$$E_T^0(s,e,\phi_1\otimes\phi_2)=\begin{cases}(\phi_1\otimes\phi_2)(0) & \text{if } T=\mathbf{0_2};\\ 0 & \text{otherwise.}\end{cases} \tag{7.4}$$

Now we calculate for $\beta=1,c$ with $c\in E$. To simplify the expressions, we ignore the term $\lambda_P(\bullet)^s$ in the following calculation. We have

$$E_T^{1,c}(s,e,\phi_1\otimes\phi_2)$$
$$=\int_{b_{11}\in F\backslash\mathbb{A}_F}\int_{b_{12}\in E\backslash\mathbb{A}_E}\int_{b_{22}\in\mathbb{A}_F}\left(\omega_\chi\left(\mathsf{w}_{2,1}m\left(\begin{pmatrix}1&\\c&1\end{pmatrix}\right)n\left(\begin{pmatrix}b_{11}&b_{12}\\b_{12}^\tau&b_{22}\end{pmatrix}\right)\right)(\phi_1\otimes\phi_2)\right)(0)$$
$$\psi\left(\mathrm{tr}\,T\begin{pmatrix}b_{11}&b_{12}\\b_{12}^\tau&b_{22}\end{pmatrix}\right)^{-1}\mathrm{d}b_{11}\mathrm{d}b_{12}\mathrm{d}b_{22}. \tag{7.5}$$

Change variables

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12}^\tau & b_{22} \end{pmatrix} \mapsto \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{12}^\tau & b_{22} \end{pmatrix} \begin{pmatrix} 1 & c^\tau \\ & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
(7.5) = &\int_{b_{11}\in F\backslash\mathbb{A}_F}\int_{b_{12}\in E\backslash\mathbb{A}_E}\int_{b_{22}\in\mathbb{A}_F}\left(\omega_\chi\left(\mathsf{w}_{2,1}n\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{12}^\tau & b_{22} \end{pmatrix}\right)m\left(\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right)\right)(\phi_1\otimes\phi_2)\right)(0) \\
&\psi\left(\operatorname{tr}\begin{pmatrix} 1 & -c^\tau \\ & 1 \end{pmatrix} T \begin{pmatrix} 1 & \\ -c & 1 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{12}^\tau & b_{22} \end{pmatrix}\right)^{-1} \mathrm{d}b_{11}\mathrm{d}b_{12}\mathrm{d}b_{22} \\
= &\int_{b_{11}\in F\backslash\mathbb{A}_F}\int_{b_{12}\in E\backslash\mathbb{A}_E}\int_{b_{22}\in\mathbb{A}_F}\left(\omega_\chi\left(\mathsf{w}_{2,1}n\left(\begin{pmatrix} 0 & 0 \\ 0 & b_{22} \end{pmatrix}\right)m\left(\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right)\right)(\phi_1\otimes\phi_2)\right)(0) \\
&\psi\left(\operatorname{tr}\begin{pmatrix} 1 & -c^\tau \\ & 1 \end{pmatrix} T \begin{pmatrix} 1 & \\ -c & 1 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{12}^\tau & b_{22} \end{pmatrix}\right)^{-1} \mathrm{d}b_{11}\mathrm{d}b_{12}\mathrm{d}b_{22}.
\end{aligned}$$

Therefore, $E_T^{1,c}(s,e,\phi_1\otimes\phi_2)=0$ unless

$$T = \begin{pmatrix} 1 & c^\tau \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$$

for some $t\in F$. In the later case,

$$\begin{aligned}
&E_T^{1,c}(s,e,\phi_1\otimes\phi_2) \\
= &\int_{b\in\mathbb{A}_F}\int_{x\in\mathbb{V}}\left(\omega_\chi\left(n\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right)m\left(\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right)\right)(\phi_1\otimes\phi_2)\right)(0,x)\psi(tb)^{-1}\mathrm{d}b\mathrm{d}x \\
= &\int_{b\in\mathbb{A}_F}\int_{x\in\mathbb{V}}\left(\omega_\chi\left(m\left(\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right)\right)(\phi_1\otimes\phi_2)\right)(0,x)\psi\left((q(x)-t)b\right)\mathrm{d}b\mathrm{d}x \\
= &\int_{b\in\mathbb{A}_F}\int_{x\in\mathbb{V}}\phi_1(cx)\phi_2(x)\psi\left((q(x)-t)b\right)\mathrm{d}b\mathrm{d}x. \qquad (7.6)
\end{aligned}$$

We remind the reader that in the above calculation, all terms involving $\lambda_P(\bullet)^s$ are ignored. In particular, when $c=0$,

$$E_T^{1,0}(s,e,\phi_1\otimes\phi_2)=\phi_1(0)\times W_t(s+\frac{1}{2},e,\phi_2). \qquad (7.7)$$

7.1.3 Holomorphic projection of analytic kernel functions

We apply 7.1.1 to the analytic kernel function $E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2))$. Here, $\phi_\alpha = \phi_\infty^0 \otimes \phi_{\alpha,\mathrm{fin}}$ is in $\mathcal{S}(\mathbb{V})$ for a totally positive definite incoherent hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank 2. Let $\chi' = \chi \mid Z'_\infty$, which is $\zeta^{\frac{\mathfrak{k}\chi}{2}}$. Then the analytic kernel function is in $\mathcal{A}(H' \times H', \chi')$. Unfortunately, it does not have the asymptotic behavior in Proposition 7.1.2. To find its holomorphic projection, we introduce the following function

$$F(s; g_1, g_2; \phi_1, \phi_2) = E(s + \frac{1}{2}, g_1, \phi_1)E(s + \frac{1}{2}, g_2, \phi_2) \in \mathcal{A}(H' \times H', \chi^\circ),$$

which is holomorphic at $s = 0$.

Proposition 7.1.5. *The difference*

$$E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)$$

has the asymptotic behavior in Proposition 7.1.2.

Proof. Since the problem is symmetric in g_1 and g_2, we prove the asymptotic behavior only in g_1. Moreover, we can assume that a_1 has 1 as its finite part. Let $a = \mathrm{diag}[a_\infty, 1]$ with $a_\infty \in E_\infty^\times$. We consider the behavior of the difference

$$E'(0, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2) - F'(0; m(a_\infty), e; \widetilde{\phi}_1, \widetilde{\phi}_2) \tag{7.8}$$

as $|a_\infty| \to \infty$, where $\widetilde{\phi}_\alpha = \omega_\chi(g_\alpha)\phi_\alpha$ for $\alpha = 1, 2$. We apply the calculation in 7.1.2 to $\widetilde{\phi}_1 \otimes \widetilde{\phi}_2$. We see that the following terms

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E_T^2(s, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2), \qquad T \in \mathrm{Her}_2(E); \\ &\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E_T^{1,c}(s, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2), \qquad c \neq 0, T \in \mathrm{Her}_2(E) \end{aligned}$$

are bounded by $O(\log |a|)$. Therefore, we only need to consider the difference

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left(E_T^0(s, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2) + \sum_{t \in F} E_{\mathrm{diag}[0,t]}^{1,0}(s, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2) - F(s; m(a_\infty), e; \widetilde{\phi}_1, \widetilde{\phi}_2) \right). \tag{7.9}$$

By (7.4) and (7.7), we have

$$\begin{aligned}
&E_T^0(s, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2) + \sum_{t \in F} E^{1,0}_{\mathrm{diag}[0,t]}(s, m(a), \widetilde{\phi}_1 \otimes \widetilde{\phi}_2) \\
&= \left(\omega_\chi(a_\infty)\widetilde{\phi}_1\right)(0)|a_\infty|^s \times \widetilde{\phi}_2(0) + \sum_{t \in F} \left(\omega_\chi(a_\infty)\widetilde{\phi}_1\right)(0)|a_\infty|^s \times W_t(s + \frac{1}{2}, e, \widetilde{\phi}_2) \\
&= \left(\omega_\chi(a_\infty)\widetilde{\phi}_1\right)(0)|a_\infty|^s \times E(s + \frac{1}{2}, e, \widetilde{\phi}_2).
\end{aligned}$$

On the other hand,

$$F(s; m(a_\infty), e; \widetilde{\phi}_1, \widetilde{\phi}_2) = \left(\left(\omega_\chi(a_\infty)\widetilde{\phi}_1\right)(0)|a_\infty|^s + \sum_{t \in F} W_t(s + \frac{1}{2}, m(a_\infty), \widetilde{\phi}_1)\right) \times E(s + \frac{1}{2}, e, \widetilde{\phi}_2).$$

Therefore,

$$(7.9) = \sum_{t \in F} W'_t(\frac{1}{2}, m(a_\infty), \widetilde{\phi}_1) \times E(\frac{1}{2}, e, \widetilde{\phi}_2) + \sum_{t \in F} W_t(\frac{1}{2}, m(a_\infty), \widetilde{\phi}_1) \times E'(\frac{1}{2}, e, \widetilde{\phi}_2),$$

which is bounded by $O(\log |a|)$. The proposition then follows. □

By the above proposition, we have

$$\begin{aligned}
&\mathrm{Pr}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\right) \\
&= \mathrm{Pr}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)\right) + \mathrm{Pr}\left(F'(0; g_1, g_2; \phi_1, \phi_2)\right) \\
&= \widetilde{\mathrm{Pr}}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)\right) + \mathrm{Pr}\left(F'(0; g_1, g_2; \phi_1, \phi_2)\right). \qquad (7.10)
\end{aligned}$$

Since

$$F'(0; g_1, g_2; \phi_1, \phi_2) = E'(\frac{1}{2}, g_1, \phi_1)E(\frac{1}{2}, g_2, \phi_2) + E(\frac{1}{2}, g_1, \phi_1)E'(\frac{1}{2}, g_2, \phi_2),$$

its holomorphic projection vanishes. Therefore,

$$\begin{aligned}
(7.10) &= \widetilde{\mathrm{Pr}}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)\right) \\
&= \widetilde{\mathrm{Pr}}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\right) - \widetilde{\mathrm{Pr}}\left(F'(0; g_1, g_2; \phi_1, \phi_2)\right) \\
&= \widetilde{\mathrm{Pr}}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\right) - \widetilde{\mathrm{Pr}}\left(E'(\frac{1}{2}, g_1, \phi_1)E(\frac{1}{2}, g_2, \phi_2)\right) - \widetilde{\mathrm{Pr}}\left(E(\frac{1}{2}, g_1, \phi_1)E'(\frac{1}{2}, g_2, \phi_2)\right).
\end{aligned}$$

But for $\{\alpha, \alpha'\} = \{1, 2\}$,

$$\begin{aligned}\widetilde{\Pr}\left(E'(\frac{1}{2}, g_\alpha, \phi_\alpha)E(\frac{1}{2}, g_{\alpha'}, \phi_{\alpha'})\right) &= \widetilde{\Pr}\left(E'(\frac{1}{2}, g_\alpha, \phi_\alpha)\right)\widetilde{\Pr}\left(E(\frac{1}{2}, g_{\alpha'}, \phi_{\alpha'})\right)\\ &= \widetilde{\Pr}\left(E'(\frac{1}{2}, g_\alpha, \phi_\alpha)\right)E_*(\frac{1}{2}, g_{\alpha'}, \phi_{\alpha'}),\end{aligned}$$

where

$$E_*(\frac{1}{2}, g_\alpha, \phi_\alpha) = \sum_{t\in F} W_t(\frac{1}{2}, g_\alpha, \phi_\alpha) = E(\frac{1}{2}, g_\alpha, \phi_\alpha) - (\omega_\chi(g_\alpha)\phi_\alpha)(0).$$

In summary, we have the following proposition.

Proposition 7.1.6. *Assume that there is a finite place $\mathfrak{p}$ such that $\phi_{\alpha,\mathfrak{p}}(0) = 0$ for $\alpha = 1, 2$. Then for $g_\alpha \in P'_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\begin{aligned}&\Pr\left(E'(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2)\right)\\ =&\widetilde{\Pr}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2)\right) - \widetilde{\Pr}\left(E'(\frac{1}{2}, g_1, \phi_1)\right)E(\frac{1}{2}, g_2, \phi_2) - E(\frac{1}{2}, g_1, \phi_1)\widetilde{\Pr}\left(E'(\frac{1}{2}, g_2, \phi_2)\right).\end{aligned}$$

7.1.4 Quasi-holomorphic projection of analytic kernel functions

We calculate $\widetilde{\Pr}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2)\right)$ for $g_\alpha \in e_{\mathsf{S}}H'(\mathbb{A}_F^{\mathsf{S}})$ under the following assumptions:

- $\phi_\alpha = \phi_\infty^0\phi_{\alpha,\mathrm{fin}}$ $(\alpha = 1, 2)$ are decomposable;
- $\phi_{1,\mathsf{S}}\otimes\phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}_{\mathsf{S}}^2)_{\mathrm{reg}}$;
- $\phi_{1,v}\otimes\phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg},d_v}$ for every $v \in \mathsf{S}$ that is nonsplit and some $d_v \geq d_{\psi_v}$ (*cf.* 2.4.2).

We recall the formula (2.23)

$$E'(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2) = \sum_{v\notin S} E'_v(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2).$$

If we apply the formula defining $\widetilde{\Pr}$ to the above summation, all terms except $E'_\iota(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2)$ for $\iota \in \Sigma_\infty$ will not change. Therefore, we fix a $\iota \in \Sigma_\infty$ and consider $\widetilde{\Pr}\left(E'_\iota(0, \imath(g_1, g_2^\vee), \phi_1\otimes\phi_2)\right)$. By Theorem 4.3.4, it amounts to consider

$$\widetilde{\Pr}\left(-2\,\mathrm{Vol}(K)\langle(Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota'}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota'})\rangle_{\mathrm{Sh}_K}\right).$$

For simplicity, we will omit the term $-2\operatorname{Vol}(K)$ in the following computation. Recall (4.56),

$$\begin{aligned}
&\left\langle (Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota'}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota'}) \right\rangle_{\mathrm{Sh}_K} \\
= &\sum_{i=1}^{l} \int_{(H(\mathbb{Q})\cap K)\backslash \mathcal{D}} \left(\sum_{x_1 \in V^n, T(x_1)\in \mathrm{Her}_n^+(E)} \left(\omega_\chi(g_1)\phi_1\right)(T(x_1), h_i^{-1}x_1) \Xi_{x_1 a(g_1)} \right) \\
&\qquad * \left(\sum_{x_2 \in V^n, T(x_2)\in \mathrm{Her}_n^+(E)} \left(\omega_\chi(g_2)\phi_2\right)(T(x_2), h_i^{-1}x_2) \Xi_{x_2 a(g_2)} \right),
\end{aligned}$$

where $a(g_\alpha)$ is denoted as a_α before. Since it is symmetric in g_1 and g_2, we only need to compute

$$\widetilde{\Pr}\left(\sum_{x\in V^n, T(x)\in \mathrm{Her}_n^+(E)} \left(\omega_\chi(\bullet)\phi\right)(T(x), x)\Xi_{xa(\bullet)} \right). \tag{7.11}$$

By Definition 7.1.3,

$$(7.11) = \sum_{t\in F^+} \widetilde{\Pr}\left(\sum_{T(x)=t} \left(\omega_\chi(\bullet)\phi\right)(t, x)\Xi_{xa(\bullet)} \right)_{\psi_t}.$$

For each term,

$$\begin{aligned}
&\widetilde{\Pr}\left(\sum_{T(x)=t} \left(\omega_\chi(\bullet)\phi\right)(t, x)\Xi_{xa(\bullet)} \right)_{\psi_t}(g) \\
= &\operatorname{const}_{s\to 0}(4\pi t) W_t^{\frac{\mathfrak{k}\chi}{2}}(g_\iota) \sum_{T(x)=t} \int_{Z'_\iota N'_\iota \backslash H'_\iota} \lambda_{P'}(h)^s \left(\omega_\chi(g^\iota h)\phi\right)(t, x)\Xi_{xa(h)} \mathrm{d}h.
\end{aligned} \tag{7.12}$$

Taking substitution $y = a\bar{a}$, we have

$$\begin{aligned}
(7.12) &= \operatorname{const}_{s\to 0}(4\pi t) W_t^{\frac{\mathfrak{k}\chi}{2}}(g_\iota) \sum_{T(x)=t} \left(\omega_\chi(g^\iota)\phi^\iota\right)(t, x) \int_0^\infty \Xi_{x\sqrt{y}} y^s \exp(-4\pi t y)\mathrm{d}y \\
&= \operatorname{const}_{s\to 0}(4\pi t) \sum_{T(x)=t} \left(\omega_\chi(g)\phi\right)(t, x) \int_0^\infty \Xi_{x\sqrt{y}} y^s \exp(-4\pi t y)\mathrm{d}y.
\end{aligned} \tag{7.13}$$

Let

$$\delta_x(z) = \frac{R(x,z)}{2t} = -\frac{(x_z, x_z)}{(x,x)}.$$

Then

$$
\begin{aligned}
(7.13) &= \mathrm{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)\int_0^\infty\left(\int_1^\infty\frac{\exp(-4\pi t y u\delta_x(z))}{u}\mathrm{d}u\right)y^s\exp(-4\pi t y)\mathrm{d}y\\
&= \mathrm{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)t^{-1-s}\int_0^\infty\int_1^\infty\frac{\exp(-4\pi y u\delta_x(z))}{u}y^s\exp(-4\pi t y)\mathrm{d}u\mathrm{d}y\\
&= \mathrm{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)t^{-1-s}\int_1^\infty\frac{1}{u}\left(\int_0^\infty\exp\left(-4\pi y(1+u\delta_x(z))\right)\right)\mathrm{d}u\\
&= \mathrm{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)\frac{\Gamma(1+s)}{(4\pi t)^{1+s}}\int_1^\infty\frac{\mathrm{d}u}{u(1+u\delta_x(z))^{1+s}}\\
&= \mathrm{const}_{s\to 1}\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)\int_1^\infty\frac{\mathrm{d}u}{u(1+u\delta_x(z))^{s}}. \qquad (7.14)
\end{aligned}
$$

Following [GZ1986], we introduce the Legendre function of the second type as follows

$$
Q_{s-1}(t)=\int_0^\infty\left(t+\sqrt{t^2-1}\cosh u\right)^{-s}\mathrm{d}u,\qquad t>1, s>0.
$$

Then the admissible Green function attached to the divisor $\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)Z_x$ is

$$
\Xi_\phi^{\mathrm{adm}}(g)_t=\mathrm{const}_{s\to 1}\,2\sum_{T(x)=t}(\omega_\chi(g)\phi)(t,x)Q_{s-1}(1+2\delta_x(z)).
$$

By [GZ1986], we have

$$
\int_1^\infty\frac{\mathrm{d}u}{u(1+uc)^s}=2Q_{s-1}(1+2c)+O(c^{-s-1}),\qquad c\to+\infty. \qquad (7.15)
$$

Combining (7.14), (7.15), Corollary 5.3.4 and Proposition 7.1.6, we have the following proposition.

Proposition 7.1.7. *Assume the three assumptions on ϕ_α at the beginning of the subsection. Then the following identity*

$$
\begin{aligned}
&\mathrm{Pr}\left(E'(0,\imath(g_1,g_2^\vee),\phi_1\otimes\phi_2)\right)\\
=&-\mathrm{Vol}(K)\sum_{v^\circ|v,v\notin\mathsf{S}}\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}-\widetilde{\mathrm{Pr}}\left(E'(\frac{1}{2},g_1,\phi_1)\right)E(\frac{1}{2},g_2,\phi_2)-E(\frac{1}{2},g_1,\phi_1)\widetilde{\mathrm{Pr}}\left(E'(\frac{1}{2},g_2,\phi_2)\right)
\end{aligned}
$$

holds for $g_\alpha\in e_{\mathsf{S}}H'(\mathbb{A}_F^{\mathsf{S}})$. In particular, the archimedean local height pairing is computed via admissible Green functions.

7.2 Proof of the main theorem

We prove the arithmetic inner product formula for $n = 1$.

7.2.1 Difference of kernel functions

We assume that

- $\phi_\alpha = \phi^0_\infty \phi_{\alpha,\mathrm{fin}}$ $(\alpha = 1, 2)$ are decomposable;
- $\phi_{\alpha,\mathsf{S}} \in \mathcal{S}(\mathbb{V}_\mathsf{S})_{\mathrm{reg}}$ $(\alpha = 1, 2)$;
- $\phi_{1,\mathsf{S}} \otimes \phi_{2,\mathsf{S}}$ is in $\mathcal{S}(\mathbb{V}^2_\mathsf{S})_{\mathrm{reg}}$;
- $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}^2_v)_{\mathrm{reg},d_v}$ for every $v \in \mathsf{S}$ that is nonsplit and some $d_v \geq d_{\psi_v}$ (*cf.* 2.4.2).

Let

$$\mathfrak{E}(g_1, g_2; \phi_1 \otimes \phi_2) = \mathrm{Pr}\left(E'(0, \imath(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\right) - \mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2),$$

which is a function in $\mathcal{A}(H' \times H', \chi')$. By (3.15), Propositions 6.1.2, 6.2.4, 6.3.4 and 7.1.7, we have that for $g_\alpha \in e_\mathsf{S} H'(\mathbb{A}^\mathsf{S}_F)$ $(\alpha = 1, 2)$, $\mathfrak{E}(g_1, g_2; \phi_1 \otimes \phi_2)$ is the sum of the following terms:

$$\begin{aligned}
\mathfrak{E}_{\mathrm{I}}(g_1, g_2; \phi_1 \otimes \phi_2) &= -E(g_1, \phi_1)A(g_2, \phi_2) - A(g_1, \phi_1)E(g_2, \phi_2) - CE(g_1, \phi_1)E(g_2, \phi_2);\\
\mathfrak{E}_{\mathrm{II}}(g_1, g_2; \phi_1 \otimes \phi_2) &= \sum_{\substack{\mathfrak{p}^\circ | \mathfrak{p} \\ \mathfrak{p} \in \mathsf{S}}} A_{(\mathfrak{p}^\circ)}(g_1, \phi_1)E(g_2, \phi_2);\\
\mathfrak{E}_{\mathrm{III}}(g_1, g_2; \phi_1 \otimes \phi_2) &= \sum_{\substack{\mathfrak{p}^\circ | \mathfrak{p} \text{ split} \\ \mathfrak{p} \in \mathsf{S}}} E_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2);\\
\mathfrak{E}_{\mathrm{IV}}(g_1, g_2; \phi_1 \otimes \phi_2) &= \sum_{\substack{\mathfrak{p}^\circ | \mathfrak{p} \text{ non-split} \\ \mathfrak{p} \in \mathsf{S}}} \theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\imath(g_1, g_2^\vee); \phi_1, \phi_2) + \theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2);\\
\mathfrak{E}_{\mathrm{V}}(g_1, g_2; \phi_1, \phi_2) &= -\mathrm{Pr}\left(E'(\frac{1}{2}, g_1, \phi_1)\right) E(\frac{1}{2}, g_2, \phi_2) - E(\frac{1}{2}, g_1, \phi_1)\mathrm{Pr}\left(E'(\frac{1}{2}, g_2, \phi_2)\right).
\end{aligned}$$

Let π be an irreducible cuspidal automorphic representation of H' such that π_∞ is a discrete series of weight $(1 - \frac{\mathfrak{k}_\chi}{2}, 1 + \frac{\mathfrak{k}_\chi}{2})$, and $\epsilon(\pi, \chi) = -1$. Then for every $f \in \pi$ and $f^\vee \in \pi^\vee$, the integral

$$\iint_{[e_\mathsf{S} H'(\mathbb{A}^\mathsf{S}_F)]^2} f(g_1) f^\vee(g_2^\vee) \chi^{-1}(\det g_2) \mathfrak{E}_\heartsuit(g_1, g_2; \phi_1 \otimes \phi_2) = 0,$$

for $\heartsuit = $ I, II, III, IV, V. This is due to the facts that each term inside $\mathfrak{E}_\heartsuit$ involves only Eisenstein series, automorphic characters or theta series in $g_\alpha \in e_\mathsf{S} H'(\mathbb{A}^\mathsf{S}_F)$ for at least one α, and that $e_\mathsf{S} H'(\mathbb{A}^\mathsf{S}_F)$

is a dense subset of $H'(F)\backslash H'(\mathbb{A}_F)$.

7.2.2 The final step

From the above subsection, we obtain the following identity:

$$\begin{aligned}
&\iint_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E'(0,\imath(g_1,g_2),\phi\otimes\phi^\vee) \\
=&\iint_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\mathbb{E}(g_1,g_2^\vee;\phi\otimes\phi^\vee) \\
=&\operatorname{Vol}(K)\iint_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\langle\Theta_\phi(g_1),\Theta_{\phi^\vee}(g_2)\rangle_{\mathrm{NT}}^K \\
=&\langle\Theta_\phi^f,\Theta_{\phi^\vee}^{f^\vee}\rangle_{\mathrm{NT}},
\end{aligned}\tag{7.16}$$

where $(\phi,\phi^\vee)$ satisfy the assumptions in the previous subsection.

Theorem 7.2.1 (Arithmetic inner product formula)**.** *Let π be an irreducible cuspidal automorphic representation of $H_1(\mathbb{A}_F)$, χ a character of $E^\times\mathbb{A}_F^\times\backslash\mathbb{A}_E^\times$, such that π_∞ is a discrete series representation of weight $(1-\frac{\mathfrak{k}^\chi}{2},1+\frac{\mathfrak{k}^\chi}{2})$, $\epsilon(\pi,\chi)=-1$. Let $\mathbb{V}$ be a totally positive definite incoherent hermitian space over $\mathbb{A}_E$ of rank 2. For each $f\in\pi$ and $\phi\in\mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty}$, we have the arithmetic theta lifting Θ_ϕ^f. Then*

1. *If $\mathbb{V}$ is not isometric to $\mathbb{V}(\pi,\chi)$, then Θ_ϕ^f is always trivial.*

2. *If $\mathbb{V}\cong\mathbb{V}(\pi,\chi)$, then for every $f\in\pi$, $f^\vee\in\pi^\vee$ and every $\phi,\phi^\vee\in\mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty}$ that are decomposable, we have*

$$\langle\Theta_\phi^f,\Theta_{\phi^\vee}^{f^\vee}\rangle_{\mathrm{NT}}=\frac{L'(\frac{1}{2},\pi,\chi)}{L_F(2)L(1,\epsilon_{E/F})}\prod_v Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee),$$

   where in the last product, almost all factors are 1.

In other words, Conjecture 3.3.6 holds when $n=1$.

Proof. We prove for (2) first, and then for (1).

1. In (2.11), we define the functional

$$\alpha(f,f^\vee,\phi,\phi^\vee)=\prod_v Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)$$

   in $\bigotimes_{v\in\Sigma}\operatorname{Hom}_{H'_v\times H'_v}(R(\mathbb{V}_v,\chi_v),\pi_v^\vee\boxtimes\chi_v\pi_v)$, which is nonzero since $\mathbb{V}\cong\mathbb{V}(\pi,\chi)$. By Proposition 2.4.10 and the assumption that π_∞ is a discrete series representation of weight $(1-\frac{\mathfrak{k}^\chi}{2},1+\frac{\mathfrak{k}^\chi}{2})$, we can choose local components f_v, $f_v^\vee$ for all $v \in \Sigma$ and ϕ_v, $\phi_v^\vee$ for $v \in \Sigma_{\mathrm{fin}}$ such that $(\phi, \phi^\vee)$ satisfy the assumptions at the beginning of the previous subsection, and $\alpha(f, f^\vee, \phi, \phi^\vee) \neq 0$.

On the other hand, we define another functional

$$\gamma(f, f^\vee, \phi, \phi^\vee) := \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{NT}} = \mathrm{Vol}(K) \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{NT}}^K,$$

which is also in $\bigotimes_{v \in \Sigma} \mathrm{Hom}_{H_v' \times H_v'}(R(\mathbb{V}_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$. By Proposition A.2.1, the space $\bigotimes_{v \in \Sigma} \mathrm{Hom}_{H_v' \times H_v'}(R(\mathbb{V}_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$ has dimension 1. In other words, γ/α is a constant. Applying (7.16) to the data we choose previously, and by (2.14), we have

$$\frac{\gamma}{\alpha} = \frac{L'(\frac{1}{2}, \pi, \chi)}{L_F(2) L(1, \epsilon_{E/F})}.$$

Therefore, (2) of the theorem follows.

2. The functional γ defined above is zero since $\mathbb{V}$ is not isometric to $\mathbb{V}(\pi, \chi)$. If we take $\phi^\vee = \overline{\phi}$ and $f^\vee = \overline{f}$, then $\Theta_\phi^f = 0$ since the pairing $\langle -, - \rangle_{\mathrm{NT}}$ is positive definite.

$\square$

Appendix

There are three appendices. In A.1, we prove the theta dichotomy for unitary groups over non-archimedean fields. For symplectic-orthogonal pairs, this is proved by S. Kudla and S. Rallis. In A.2, we prove a multiplicity one result in the theory of local theta correspondence in a special case. In A.3, we study the theta correspondence of unramified representations for unitary groups, following the method of S. Rallis.

A.1 Theta dichotomy for unitary groups

In this section, we prove the theta dichotomy for unitary groups over non-archimedean fields[1]. For symplectic-orthogonal pairs, this is proved by Kudla–Rallis in [KR2005]. We will follow the same line.

Let F be a non-archimedean local field whose characteristic is different from 2, and a quadratic field extension E/F with $\mathrm{Gal}(E/F) = \{1, \tau\}$. Let $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) be the ring of integers, ϖ (resp. ϖ_E) a uniformizer of F (resp. E), and $q = |\mathcal{O}_F/\varpi\mathcal{O}_F|$ (resp. $q_E = |\mathcal{O}_E/\varpi_E\mathcal{O}_E|$).

Let $n \geq 1$ be an integer. We denote by $V_n^{\pm}$ the (left) hermitian spaces over E of dimension n such that $\epsilon(V_n^{\pm}) = \pm 1$, and $H_n^{\pm} = \mathrm{U}(V_n^{\pm})$. Let $(W, \langle -, - \rangle)$ be a (right) skew-hermitian space over E of the same dimension n, and $H' = \mathrm{U}(W)$.

As in 2.2.1, we have the doubling space $W \oplus (-W)$, and $H'' = \mathrm{U}(W \oplus (-W))$, where $(-W) = (W, -\langle -, - \rangle)$. We fix a character $\chi : E^\times \to \mathbb{C}^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^n$. Let $I_n(s, \chi)$ be the space of degenerate principal series, which is an admissible representation of H''. We have $I_n(0, \chi) = R(V_n^+, \chi) \oplus R(V_n^-, \chi)$. For any irreducible admissible representation π of H',

$$\mathrm{Hom}_{H' \times H'}\left(I_n(0, \chi), \pi \boxtimes \chi\pi^\vee\right) \neq 0.$$

In fact, we have the following result of *theta dichotomy*:

Proposition A.1.1. *There is exactly one between the following two spaces*

$$\mathrm{Hom}_{H' \times H'}(R(V_n^+, \chi), \pi \boxtimes \chi\pi^\vee); \quad \mathrm{Hom}_{H' \times H'}(R(V_n^-, \chi), \pi \boxtimes \chi\pi^\vee),$$

which is nonzero.

Proof. From the discussion above, we know that there is at least one space that is nonzero. If they are both nonzero, then $\mathrm{Hom}_{H'}(\mathcal{S}((V_n^+)^n), \pi) \neq 0$ and $\mathrm{Hom}_{H'}(\mathcal{S}((V_n^-)^n), \pi) \neq 0$, where H' acts on both spaces of Schwartz functions through the Weil representation ω_χ.

Let $g_0 \in \mathrm{GL}_F(W)$ be a τ-linear automorphism such that

$$\langle w_1 g_0, w_2 g_0 \rangle = \langle w_1, w_2 \rangle$$

for all $w_1, w_2 \in W$. By [MVW1987],

$$\pi \circ \mathrm{Ad}\, g_0 \cong \pi^\vee,$$

[1] During the preparation of this article, Gong and Grenié [GG2011] also prove the same results independently. Our methods are similar and we refer to their article for more details.

and we have

$$\left(\omega_\chi \circ \operatorname{Ad} g_0, \mathcal{S}((V_n^-)^n)\right) \cong \left(\omega_{\chi^{-1}}, \mathcal{S}((-V_n^-)^n)\right).$$

Therefore,

$$\operatorname{Hom}_{H'}\left((\omega_\chi, \mathcal{S}((V_n^+)^n)), \pi\right) \neq 0;$$
$$\operatorname{Hom}_{H'}\left((\omega_{\chi^{-1}}, \mathcal{S}((-V_n^-)^n)), \pi^\vee\right) \neq 0.$$

Taking product and the pairing between π and $\pi^\vee$, we have

$$\operatorname{Hom}_{H'}\left((\omega_1, \mathcal{S}((V_n^+ \oplus (-V_n^-))^n)), \mathbf{1}\right) \neq 0,$$

where $\mathbf{1}$ is the trivial representation of H'. It is easy to see that $V_n^+ \oplus (-V_n^-) \cong V_{2n}^-$. We have $\operatorname{Hom}_{H'}(\mathcal{S}((V_{2n}^-)^n), \mathbf{1}) \neq 0$. By Lemma A.1.2 below, we have

$$\operatorname{Hom}_{H'\times H'}(R_n(V_{2n}^-, 1), \mathbf{1}) \neq 0.$$

Then by [KS1997, Theorem 1.2 (4)], $R_n(V_{2n}^-, 1)$ is the unique irreducible submodule of $I_n(\frac{n}{2}, 1)$, which is also the image of the intertwining map

$$M^*(-\frac{n}{2}, 1) : I_n(-\frac{n}{2}, 1) \to R_n(V_{2n}^-, 1).$$

By [KS1997, Proposition 4.3 (2)], $\ker(M^*(-\frac{n}{2}, 1)) = \mathbb{C} \cdot \varphi^0$, where $\varphi^0 \in I_n(-\frac{n}{2}, 1)$ is the constant function with value 1. Therefore, there is a nonzero functional

$$\zeta \in \operatorname{Hom}_{H'\times H'}\left(I_n(-\frac{n}{2}, 1), \mathbf{1}\right)$$

such that $\zeta(\varphi^0) = 0$. By Lemma A.1.3 below,

$$\operatorname{Hom}_{H'\times H'}\left(I_n(-\frac{n}{2}, 1), \mathbf{1}\right) = \mathbb{C} \cdot Z(-\frac{n}{2}, 1, 1, -)$$

and $Z(-\frac{n}{2}, 1, 1, \varphi^0) \neq 0$, which is a contradiction. The proposition is proved. □

Lemma A.1.2. *For any hermitian space V over E of dimension m and χ such that $\chi|_{F^\times} = \epsilon_{E/F}^m$, the following are equivalent:*

- $\mathrm{Hom}_{H'}\left((\omega_\chi, \mathcal{S}(V^n)), \pi\right) \neq 0$;

- $\mathrm{Hom}_{H' \times H'}(R_n(V, \chi), \pi \boxtimes \chi\pi^\vee) \neq 0$.

Proof. Since $\mathrm{Hom}_{H'}\left((\omega_\chi, \mathcal{S}(V^n)), \pi\right) \neq 0$ is equivalent to $\Theta_\chi(\pi, V) \neq 0$, and $\Theta_\chi(\pi, V)$ admits an irreducible quotient by [MVW1987, Chapitre 3, IV, Theorème 4 (2-a)], the lemma follows by [HKS1996, Proposition 3.1]. $\square$

Before we proceed, let us recall some notations from [HKS1996, Section 4]. Let $Y \subset W \oplus (-W)$ be the graph of the identity map and $P_Y \subset H''$ the parabolic subgroup that stabilizes Y. Let r_0 be the Witt index of W. Then $P_Y \backslash H''$ can be canonically identified with the set of isotropic n-dimensional subspaces of $W \oplus (-W)$. The $H' \times H'$-orbit of Z in $P_Y \backslash H''$ is uniquely determined by

$$d = \dim(Z \cap W) = \dim(Z \cap (-W)).$$

Therefore,

$$H'' = \coprod_{0 \leq d \leq r_0} \Omega_d := \coprod_{0 \leq d \leq r_0} P_Y \delta_d (H' \times H')$$

for some representative $\delta_d \in H''$. The (topological) closure

$$\overline{\Omega_r} = \coprod_{r' \geq r} \Omega_{r'}.$$

Let

$$I_n(s, \chi) = I_n^{(r_0)}(s, \chi) \supset I_n^{(r_0 - 1)}(s, \chi) \supset \cdots \supset I_n^{(0)}(s, \chi)$$

be the filtration given by support, and

$$Q_n^{(r)} = I_n^{(r)}(s, \chi) \supset I_n^{(r-1)}(s, \chi) \cong \mathrm{Ind}_{P_r \times P_r}^{H' \times H'} \left(\chi|\bullet|_E^{s + \frac{r}{2}} \boxtimes \chi|\bullet|_E^{s + \frac{r}{2}} \otimes \mathcal{S}(H'_{n-2r})\right),$$

where the induction is normalized. Here, P_r is the parabolic subgroup of H' with the Levi quotient M_r that is isomorphic to $\mathrm{GL}_r(E) \times H'_{n-2r}$, and the maximal unipotent subgroup N_r, where H'_{n-2r} is the unitary group of some skew-hermitian space of dimension $n - 2r$.

For every section $\varphi_s \in I_n(s, \chi)$ and matrix coefficient ϕ of $\pi^\vee$, we define the zeta integral

$$Z(s, \chi, \phi, \varphi) = \int_{H'} \phi(g) \varphi_s((g, 1)) \mathrm{d}g.$$

If the section is standard, it has a meromorphic continuation to the entire complex. If for some $s_0 \in \mathbb{C}$

at which every such continuation is holomorphic, then the zeta integral defines a nonzero element

$$Z(s_0,\chi,-,-)\in \operatorname{Hom}_{H'\times H'}(I_n(s_0,\chi),\pi\boxtimes\chi\pi^\vee).$$

Moreover, if $\operatorname{Hom}_{H'\times H'}(Q_r(s_0,\chi),\pi\boxtimes\chi\pi^\vee)=0$ for every $r\geq 1$, then all zeta integrals are holomorphic at s_0, and

$$\operatorname{Hom}_{H'\times H'}(I_n(s_0,\chi),\pi\boxtimes\chi\pi^\vee)=\mathbb{C}\cdot Z(s_0,\chi,-,-).$$

For the trivial representation, we have the following lemma.

Lemma A.1.3. *For $\chi=1$ and $s_0=-\frac{n}{2}$, $\operatorname{Hom}_{H'\times H'}(Q_r(s_0,1),\mathbf{1})=0$ for every $r\geq 1$. In particular, $Z(s_0,1,1,\varphi^0)\neq 0$.*

Proof. For $r\geq 1$ and every irreducible admissible representation π of H',

$$\begin{aligned}
&\operatorname{Hom}_{H'\times H}(Q_r(s,1),\pi\boxtimes\pi^\vee)\\
&=\operatorname{Hom}_{H'\times H'}\left(\pi^\vee\boxtimes\pi,\operatorname{Ind}_{P_r\times P_r}^{H'\times H'}(|\bullet|_E^{-s-\frac{r}{2}}\boxtimes|\bullet|_E^{-s-\frac{r}{2}}\otimes\mathscr{C}^\infty(H'_{n-2r}))\right)\\
&=\operatorname{Hom}_{M_r\times M_r}\left((\pi^\vee)_{N_r}\boxtimes\pi_{N_r},|\bullet|_E^{-s+\frac{n}{2}-r}\boxtimes|\bullet|_E^{-s+\frac{n}{2}-r}\otimes\mathscr{C}^\infty(H'_{n-2r})\right).
\end{aligned}$$

In particular, when $\pi=\mathbf{1}$, we have

$$\begin{aligned}
&\operatorname{Hom}_{H'\times H}(Q_r(s,1),\mathbf{1})\\
&=\operatorname{Hom}_{M_r\times M_r}\left(\mathbf{1},|\bullet|_E^{-s+\frac{n}{2}-r}\boxtimes|\bullet|_E^{-s+\frac{n}{2}-r}\otimes\mathscr{C}^\infty(H'_{n-2r})\right)\\
&=\operatorname{Hom}_{\mathrm{GL}_r(E)\times\mathrm{GL}_r(E)}\left(\mathbf{1},|\bullet|_E^{-s+\frac{n}{2}-r}\boxtimes|\bullet|_E^{-s+\frac{n}{2}-r}\right).
\end{aligned}$$

Therefore, if $s_0=-\frac{n}{2}$, then $\operatorname{Hom}_{H'\times H'}(Q_r(s_0,1),\mathbf{1})=0$ for every $r\geq 1$. The fact $Z(s_0,1,1,\varphi^0)\neq 0$ is due to the following lemma. $\square$

Lemma A.1.4. *Let φ_s^0 be the standard section such that $\varphi_s^0|_{\mathcal{K}}=1$, where $\mathcal{K}$ is the maximal compact subgroup of H''. Let $b_n(s)=\prod_{i=0}^{n-1}L(2s+n-i,\epsilon_{E/F}^i)$. Then we have three cases:*

1. *If $n=2m$ $(m\geq 1)$ and $r_0=m$,*

$$Z(s,1,1,\varphi^0)=\frac{1}{b_n(s)}\prod_{i=-m}^{m-1}\left(1-q_E^{i-s}\right)^{-1}.$$

2. *If $n = 2m$ $(m \geq 1)$ and $r_0 = m-1$,*

$$Z(s,1,1,\varphi^0) = \frac{1}{b_n(s)} \left(1 - q_E^{-s}\right) \prod_{i=-m}^{m-1} \left(1 - q_E^{i-s}\right)^{-1}.$$

3. *If $n = 2m+1$ $(m \geq 0)$ and $r_0 = m$,*

$$Z(s,1,1,\varphi^0) = \frac{1}{b_n(s)} \prod_{i=-m}^{m} \left(1 - q_E^{i-s-\frac{1}{2}}\right)^{-1}.$$

Proof. For (1) and (3), if E/F is unramified, they are [GPSR1987, Part A, Proposition 6.2] (for n even) and [Li1992, Theorem 3.1] (for n odd). The calculation for the integral

$$\int_{H'} \varphi_s^0((g,1))\mathrm{d}g$$

in these papers works also for E/F ramified, since the only thing we need to check is the ratio $c_w(\chi)$ of the intertwining operator (*cf.* [Li1992, Page 189]). The formula for such ratio holds more generally, in particular, when E/F is ramified, by [Cas1980, Theorem 3.1].

For (2), one can prove similarly as in [Li1992, Section 3]. In fact, (1) is enough for our use. □

A.2 Uniqueness of local invariant functionals

In this section, we fix a place $v \in \Sigma$ and suppress it from notations. We prove that the space $\mathrm{Hom}_{H' \times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi)$ is of dimension 1, following [HKS1996]. Here, χ is a character of $E^\times$. If χ is trivial on $F^\times$, *i.e.* $\chi^{-1} = \chi^\tau$, then we define a character π_χ of H' as follows. For every element $g \in H'$, $\det g$ is in $E^{\times,1}$. In particular, $\det g = e_g/e_g^\tau$ for some $e_g \in E^\times$ by Hilbert's Theorem 90. Set $\pi_\chi(g) = \chi(e_g)$, which is independent of e_g we choose.

Proposition A.2.1. *Let χ be a character of $E^\times$, and π an irreducible admissible representation of H' that is not π_χ^{-1} if $\chi^{-1} = \chi^\tau$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H' \times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi) = 1,$$

and $L(s,\pi,\chi)$ is holomorphic at $s = \frac{1}{2}$.

In fact, the proposition holds for arbitrary representation π. Since we do not need such generality in this article, we keep the assumption in the previous proposition for simplicity. In what follows, we also assume that v is finite for simplicity. The case for archimedean places is similar, and actually will not be used in the article.

Recall from 2.2.1, we have, as a special case, the following decomposition of double cosets $H'' = \Omega_0 \coprod \Omega_1$, where $\Omega_0 = P\gamma_0\imath(H' \times H')$ is open and $\Omega = P\imath(H' \times H')$ is closed. Therefore, we have a subspace $I_2^{(0)}(0,\chi) \subset I_2(0,\chi)$, where

$$I_2^{(0)}(0,\chi) = \{\varphi \in I_2(0,\chi) \mid \operatorname{Supp}\varphi \subset \Omega_0\},$$

which is invariant under the action of $H' \times H'$ by right translation via $\imath$. As $H' \times H'$-representations, we denote

$$Q_2^{(0)}(0,\chi) = I_2^{(0)}(0,\chi); \qquad Q_2^{(1)}(0,\chi) = I_2(0,\chi)/I_2^{(0)}(0,\chi).$$

We have a linear isomorphism

$$\begin{aligned} Q_2^{(0)}(0,\chi) &\to \mathcal{S}(H')(1 \otimes \chi) \\ \varphi &\mapsto \Psi(g) = \varphi\left(\gamma_0\imath(g, \mathbf{1}_2)\right), \end{aligned}$$

where $\mathcal{S}(H)$ denotes the space the Schwartz functions on H', which is viewed as a representation of $H' \times H'$. Since

$$\varphi\left(\gamma_0\imath(g,\mathbf{1}_2)\imath(g_1,g_2)\right) = \varphi\left(\gamma_0\imath(g_2,g_2)\imath(g_2^{-1}gg_1,\mathbf{1}_2)\right) = \chi(\det g_2)\varphi\left(\gamma_0\imath(g_2^{-1}gg_1,\mathbf{1}_2)\right),$$

the above isomorphism is $H' \times H'$-equivariant. There is a unique, up to constant, $H' \times H'$-invariant functional on $\mathcal{S}(H') \otimes (\pi \boxtimes \pi^\vee)$ given by

$$\Psi \otimes (f \otimes f^\vee) \mapsto \int_{H'} \langle \pi(g)f, f^\vee \rangle \Psi(g) \mathrm{d}g.$$

Since

$$\begin{aligned} &\operatorname{Hom}_{H' \times H'}\left(\mathcal{S}(H') \otimes (\pi \boxtimes \pi^\vee), \mathbb{C}\right) = \operatorname{Hom}_{H' \times H'}\left(\mathcal{S}(H'), \pi^\vee \boxtimes \pi\right) \\ &= \operatorname{Hom}_{H' \times H'}\left(\mathcal{S}(H') \otimes (1 \boxtimes \chi), \pi^\vee \boxtimes \chi\pi\right) = \operatorname{Hom}_{H' \times H'}\left(Q_2^{(0)}(0,\chi), \pi^\vee \boxtimes \chi\pi\right), \end{aligned}$$

we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_{H' \times H'}\left(Q_2^{(0)}(0,\chi), \pi^\vee \boxtimes \chi\pi\right) = 0.$$

For $Q_2^{(1)}(0,\chi)$, we have the following lemma.

Lemma A.2.2. *If $\chi^{-1} \neq \chi^\tau$, or $\chi^{-1} = \chi^\tau$ but $\pi \neq \pi_\chi^{-1}$, then*

$$\mathrm{Hom}_{H' \times H'}\left(Q_2^{(1)}(0,\chi), \pi^\vee \boxtimes \chi\pi\right) = 0.$$

Proof. It is easy to see that the following map

$$\begin{aligned} Q_2^{(1)}(0,\chi) &\to I_1(\frac{1}{2},\chi) \boxtimes I_1(\frac{1}{2},\chi) \\ \bar{\varphi} &\mapsto ((g_1,g_2) \mapsto \bar{\varphi}\left(\imath(g_1,g_2)\right)) \end{aligned}$$

is an $H' \times H'$-equivariant isomorphism. Therefore,

$$\begin{aligned} &\mathrm{Hom}_{H' \times H'}\left(Q_2^{(1)}(0,\chi), \pi^\vee \boxtimes \chi\pi\right) \\ =\ &\mathrm{Hom}_{H' \times H'}\left(I_1(\frac{1}{2},\chi) \boxtimes I_1(\frac{1}{2},\chi), \pi^\vee \boxtimes \chi\pi\right) \\ =\ &\mathrm{Hom}_{H' \times H'}\left(\pi \boxtimes \chi^{-1}\pi^\vee, I_1(-\frac{1}{2},\chi^{-1}) \boxtimes I_1(-\frac{1}{2},\chi^{-1})\right). \end{aligned}$$

By [KS1997, Theorem 1.2] for v finite and nonsplit, [KS1997, Theorem 1.3] for v finite and split (and [Lee1994, Theorem 6.10 (1-b)] for v infinite), the only (possible) irreducible H'-submodule properly contained in $I_1(-\frac{1}{2},\chi^{-1})$ is isomorphic to π_χ^{-1}. Therefore, the lemma follows by our assumption. □

Proof of Proposition A.2.1. The normalized zeta integral (2.8) has already defined a nonzero element in $\mathrm{Hom}_{H' \times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi)$. Therefore, the dimension is at least 1. If it is greater than one, we can find a nonzero element in $\mathrm{Hom}_{H' \times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi)$ whose restriction to $I_2^{(0)}(0,\chi)$ is zero since $\dim_{\mathbb{C}} \mathrm{Hom}_{H' \times H'}\left(Q_2^{(0)}(0,\chi), \pi^\vee \boxtimes \chi\pi\right) = 1$. Then it defines a nonzero element in $\mathrm{Hom}_{H' \times H'}\left(Q_2^{(1)}(0,\chi), \pi^\vee \boxtimes \chi\pi\right)$ that is 0 by the above lemma. Therefore,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H' \times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi) = 1.$$

For the L-factor, the restriction of the normalized zeta integral to $I_2^{(0)}(0,\chi)$ is nonzero. Since the original zeta integral is absolutely convergent at $s=0$ if $\varphi \in I_2^{(0)}(0,\chi)$, $L(s,\pi,\chi)$ can not have a pole at $s=\frac{1}{2}$, by realizing that $b_2(s)$ is holomorphic and nonzero at $s=0$. □

A.3 Theta correspondence of unramified representations

In this section, we study the theta correspondence of unramified representations for unitary groups.

Let F be a p-adic local field with $p \neq 2$, E/F an unramified quadratic field extension with $\mathrm{Gal}(E/F) = \{1, \tau\}$. Let $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) be the ring of integers of F (resp. E), ϖ a uniformizer of $\mathcal{O}_F$, and q the cardinality of $\mathcal{O}_F/\varpi\mathcal{O}_F$. Let ψ be an unramified additive character of F, which determines an unramified additive character of E by composing with $\frac{1}{2}\operatorname{Tr}_{E/F}$. Let $\mathrm{d}x$ be the selfdual Haar measure of E with respect to $\psi \circ \left(\frac{1}{2}\operatorname{Tr}_{E/F}\right)$, and $\mathrm{d}^\times x = \frac{\mathrm{d}x}{|x|_E}$ the Haar measure of $E^\times$, normalized such that $|\varpi|_E = q^{-2}$. We will use slightly different notations from 2.1.1.

Let $n, m \geq 1$ be two integers, and $r = \min\{m, n\}$. Let $(W_n, \langle -, - \rangle)$ be a skew-hermitian space over E whose skew hermitian form is given by

$$\begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}$$

under a basis $\{e_1, \ldots, e_n; e_1^*, \ldots, e_n^*\}$. Let $(V_m, (-, -))$ be a hermitian space over E whose hermitian form is given by

$$\begin{pmatrix} & \mathbf{1}_m \\ \mathbf{1}_m & \end{pmatrix}$$

under a basis $\{f_1, \ldots, f_m; f_1^*, \ldots, f_m^*\}$. Let $H'_n = \mathrm{U}(W_n)$ (resp. $H_m = \mathrm{U}(V_n)$) be the corresponding group of isometries, and

$$K'_n = H'_n \cap \mathrm{GL}(\mathcal{O}_E\langle e_1, \ldots, e_n; e_1^*, \ldots, e_n^* \rangle) \text{ (resp. } K_m = H_m \cap \mathrm{GL}(\mathcal{O}_E\langle f_1, \ldots, f_m; f_1^*, \ldots, f_n^* \rangle))$$

be a hyperspecial maximal subgroup. We have a Weil representation $\omega = \omega_{\chi=1,\psi}$ on the space $\mathcal{S}(V_m^n)$ of Schwartz functions, whose formulae are given in 2.1.1.

Let $W_{n,i}^* = \operatorname{span}_E\{e_{i+1}^*, \ldots, e_n^*\}$ for $0 \leq i \leq n$, and $V_{m,j}^* = \operatorname{span}_E\{f_{j+1}^*, \ldots, f_m^*\}$ for $0 \leq j \leq m$. Then we have filtration of the maximal isotropic subspaces $W_{n,0}^*$ and $V_{m,0}^*$, respectively as

$$W_{n,0}^* \supset W_{n,1}^* \supset \cdots \supset W_{n,n}^* = \{0\}; \qquad V_{m,0}^* \supset V_{m,1}^* \supset \cdots \supset V_{m,m}^* = \{0\}.$$

Up to conjugacy, the maximal parabolic subgroups of $H'_n \times H_m$ are precisely those subgroups $P'_{n,i} \times P_{m,j}$ that consists of elements (h', h) stabilizing the subspace $W_{n,i}^* \otimes V_{m,j}^* \subset W_n \otimes V_m$, for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$. Let $N'_{n,i} \times N_{m,j}$ be its unipotent radical. Then the Levi quotient $P'_{n,i} \times P_{m,j}/N'_{n,i} \times$

$N_{m,j}$ is isomorphic to $(\mathrm{GL}_{n-i}(E)\times H'_i)\times(\mathrm{GL}_{m-j}(E)\times H_j)$. For $0\le t\le n$, we define a Zariski closed subsets Σ_t to be

$$\Sigma_t=\{x=(x_1,\dots,x_n)\in V_m^n \mid (x_i,x_j)=0 \text{ for } t+1\le j\le n\}.$$

We say that a function $\phi\in\mathcal{S}(V_m^n)$ is *spherical* if it is invariant under the action of $K'_n\times K_m$. The same proof of [Ral1984, Proposition 2.2] implies the following lemma.

Lemma A.3.1. *Let ϕ be a spherical function in $\mathcal{S}(V_m^n)$ such that for every $h'\in H'_n$, $\omega(h')\phi$ vanishes on the subset Σ_0. Then $\omega(h')\phi$ vanishes identically.*

We identify V_m^n with $\mathrm{Mat}_{2m,n}(E)$ via the basis $\{f_1,\dots,f_m;f_1^*,\dots,f_m^*\}$. Then the action of $\mathrm{GL}_n(E)\times H_m$ on V_m^n is given by $(A,h).X=hXA^{-1}$. We have the following version of [Ral1984, Lemma 3.1],

Lemma A.3.2. *Let $\Sigma_0^{(i)}=\{X\in\Sigma_0\mid \mathrm{rank}\,X=i\}$. Then $\Sigma_0^{(i)}$, if nonempty, is an orbit under the action of $\mathrm{GL}_n(E)\times H_m$; and Σ_0 is a disjoint union of orbits of the form $\Sigma_0^{(i)}$ for $i=0,1,\dots,r$, in which $\Sigma_0^{(r)}$ is the unique open one.*

Let

$$B'_n=\left\{\begin{pmatrix} A & \\ & {}^tA^{\tau,-1}\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & B\\ & \mathbf{1}_n\end{pmatrix} \mid A \text{ is lower triangular and } B\in\mathrm{Her}_n(E)\right\},$$

whose Levi decomposition is $B'_n=T'_nU'_n$ with

$$T'_n=\{\mathrm{diag}[t_1,\dots,t_n;t_1^{\tau,-1},\dots,t_n^{\tau,-1}]\mid t_i\in E^\times\},$$

and U'_n being the unipotent radical. Let

$$B_{m,r}=\left\{\begin{pmatrix} A & \\ & {}^tA^{\tau,-1}\end{pmatrix}\begin{pmatrix}\mathbf{1}_m & B\\ & \mathbf{1}_m\end{pmatrix} \mid A=\begin{pmatrix}A_1 & A_2\\ & A_3\end{pmatrix}\right\},$$

where $A_1\in\mathrm{Mat}_{r,r}(E)$ is lower-triangular, $A_3\in\mathrm{Mat}_{m-r,m-r}(E)$ is upper-triangular, $A_2\in\mathrm{Mat}_{r,m-r}(E)$, and B is skew-hermitian. We have the Levi decomposition $B_{m,r}=T_mU_{m,r}$ with $T_m=T'_m$ and $U_{m,r}$ being the unipotent radical. Then $B'_n\times B_{m,r}$ is a minimal parabolic subgroup of $H'_n\times H_m$.

Review some facts about spherical representations of $H'_n\times H_m$. For $\nu=(\nu_1,\dots,\nu_n)\in\mathbb{C}^n$, define the space $I'(\nu)$, which consists of all locally constant functions $\varphi : H'_n \to \mathbb{C}$ satisfying

$$\varphi(h't'u') = \delta_n'^{-\frac{1}{2}}(t') \prod_{i=1}^{n} |t'_i|_E^{\nu_i}\, \varphi(h')$$

for all $h' \in H'_n$, $t \in T'_n$, and $u' \in U'_n$. Here,

$$\delta'_n = \prod_{i=1}^{n} |t'_i|_E^{2i-1}$$

is the modulus function of B'_n. These $I'(\nu)$ provide all spherical principal series of H'_n. Let $\mathcal{S}(H'_n//K'_n)$ be the spherical Hecke algebra of H'_n. We have the Fourier–Satake isomorphism

$$\mathrm{FS} : \mathcal{S}(H'_n//K'_n) \to \mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]^{W(H'_n)},$$

such that for every $f' \in \mathcal{S}(H'_n//K'_n)$,

$$\mathrm{FS}(f')(q^{2\nu_1}, q^{-2\nu_1}, \dots, q^{2\nu_n}, q^{-2\nu_n}) = \mathrm{trace}_{I'(\nu)}(f').$$

For $\mu = (\mu_1, \dots \mu_m) \in \mathbb{C}^m$, define the space $I(\mu)$, which consists of all locally constant functions $\varphi : H_m \to \mathbb{C}$ satisfying

$$\varphi(htu) = \delta_{m,r}^{-\frac{1}{2}}(t) \prod_{j=1}^{m} |t_j|_E^{\mu_j}\, \varphi(h)$$

for all $h \in H_m$, $t \in T_m$, and $u \in U_{m,r}$. Here,

$$\delta_{m,r}(t) = \prod_{j=1}^{r} |t_j|_E^{2m-2r+2j-1} \prod_{j=r+1}^{m} |t_j|_E^{2m-2j+1}$$

is the modulus function of $B_{m,r}$. These $I(\mu)$ provide all spherical principal series of H_m. Let $\mathcal{S}(H_m//K_m)$ be the spherical Hecke algebra of H_m. We have the Fourier–Satake isomorphism

$$\mathrm{FS} : \mathcal{S}(H_m//K_m) \to \mathbb{C}[X_1, X_1^{-1}, \dots, X_m, X_m^{-1}]^{W(H_m)},$$

such that for every $f \in \mathcal{S}(H_m//K_m)$,

$$\mathrm{FS}(f)(q^{2\mu_1}, q^{-2\mu_1}, \dots, q^{2\mu_m}, q^{-2\mu_m}) = \mathrm{trace}_{I(\mu)}(f).$$

Let $Y_r \subset \mathrm{GL}_r(E)$ be the group of lower-triangular matrices, which has the Levi decomposition

$Y_r = A_r L_r$. Here, $A_r = \{\mathrm{diag}[a_1, \ldots, a_r] | a_i \in E^\times\}$ and L_r is the unipotent radical. We write elements in L_r in the form $(l_{jk})_{j>k}$. The group Y_r has the following right invariant measure

$$\mathrm{d}y_r = \prod_{i=1}^{r} |a_i|_E^{2i-(r+1)} \mathrm{d}^\times a_i \prod_{1 \le k < j \le r} \tilde{\mathrm{d}} l_{jk},$$

where $\tilde{\mathrm{d}} l_{jk}$ is certain measure on L_r normalized as in [Ral1982, Page 490]. Let $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r$ such that $\mathrm{Re}\,\sigma_i \gg 0$. For every function $\phi \in \mathcal{S}(V_m^n)$, the following integral

$$Z_\sigma(\phi) = \int_{Y_r} \phi\left(\begin{pmatrix} y_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) \prod_{i=1}^{r} |a_i|_E^{\sigma} \, \mathrm{d}y_r$$

is absolutely convergent. Define a map $\mathcal{Z}_\sigma$ sending $\phi \in \mathcal{S}(V_m^n)$ to the function

$$(h', h) \mapsto Z_\sigma\left(\omega(h'^{-1}, h^{-1})\phi\right)$$

on $H_n' \times H_m$. It is a nonzero $H_n' \times H_m'$-equivariant map from $\mathcal{S}(V_m^n)$ to $\mathcal{S}(H_n' \times H_m)$.

Lemma A.3.3. *For $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r$ such that $\mathrm{Re}\,\sigma_i \gg 0$, the image of the above intertwining map $\mathcal{Z}_\sigma$ lies in $I'(\nu) \otimes I(\mu)$, where*

$$\nu = \left(2 + \sigma_1 - m - \frac{3}{2}, \ldots, 2r + \sigma_r - m - \frac{3}{2}, (r+1) - m - \frac{1}{2}, \ldots, n - m - \frac{1}{2}\right);$$
$$\mu = \left(-2 - \sigma_1 + m + \frac{3}{2}, \ldots, -2r + \sigma_r + m + \frac{3}{2}, -(r+1) + m + \frac{1}{2}, \ldots, \frac{1}{2}\right).$$

Proof. We have

$$\begin{aligned} \mathcal{Z}_\sigma(\phi)(h't'u', htu) &= \int_{Y_r} \left(\omega\left(u'^{-1}t'^{-1}h'^{-1}, u^{-1}t^{-1}h^{-1}\right)\phi\right)\left(\begin{pmatrix} y_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) \prod_{i=1}^{r} |a_i|_E^{\sigma} \mathrm{d}y_r \\ &= \int_{Y_r} \left(\omega\left(t'^{-1}h'^{-1}, t^{-1}h^{-1}\right)\phi\right)\left(\begin{pmatrix} y_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) \prod_{i=1}^{r} |a_i|_E^{\sigma} \mathrm{d}y_r \\ &= \int_{Y_r} |\det t'|_E^{-m} \left(\omega\left(h'^{-1}, h^{-1}\right)\phi\right)\left(t\begin{pmatrix} y_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} t'^{-1}\right) \prod_{i=1}^{r} |a_i|_E^{\sigma} \mathrm{d}y_r. \end{aligned} \tag{A.1}$$

Changing variable $y_r \mapsto y''_r = \mathrm{diag}[t_1, \ldots, t_r]y_r$, we have

$$\prod_{i=1}^{r} |a_i|_E^{\sigma} \mathrm{d}y_r = \prod_{i=1}^{r} |t_i|_E^{r-3i-\sigma_i+2} \prod_{i=1}^{r} |t_i a_i|_E^{\sigma} \mathrm{d}y''_r.$$

Changing variable $y_r \mapsto y'_r = \mathrm{diag}[t_1'^{-1}, \ldots, t_r'^{-1}]$, we have

$$\prod_{i=1}^{r} |a_i|_E^{\sigma} \mathrm{d}y_r = \prod_{i=1}^{r} |t_i|_E^{i+\sigma_i-1} \prod_{i=1}^{r} \left|t_i'^{-1} a_i\right|_E^{\sigma} \mathrm{d}y'_r.$$

Therefore,

$$\begin{aligned}(\text{A.1}) &= \prod_{i=1}^{r} |t'_i|_E^{i+\sigma_i-m-1} \prod_{i=r+1}^{n} |t'_i|_E^{-m} \prod_{j=1}^{r} |t_j|_E^{r-3j-\sigma_j+2} \int_{Y_r} \left(\omega\left(h'^{-1}, h^{-1}\right)\phi\right) \left(\begin{pmatrix} y_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) \prod_{i=1}^{r} |a_i|_E^{\sigma} \mathrm{d}y_r \\ &= \prod_{i=1}^{r} |t'_i|_E^{i+\sigma_i-m-1} \prod_{i=r+1}^{n} |t'_i|_E^{-m} \prod_{j=1}^{r} |t_j|_E^{r-3j-\sigma_j+2} \left(\mathcal{Z}_\sigma(\phi)\right)(h', h).\end{aligned}$$

The lemma follows immediately. □

The above lemma implies the following facts. If $m \geq n = r$, then there is a surjective homomorphism

$$\Phi_{m,n} : \mathcal{S}(H_m//K_m) \to \mathcal{S}(H'_n//K'_n)$$

satisfying that

$$\mathcal{Z}_\sigma \circ (\Phi_{m,n}(f) - f) = 0$$

for all $f \in \mathcal{S}(H_m//K_m)$ and $\mathrm{Re}\,\sigma_i \gg 0$. Using the Fourier–Satake isomorphism, the map $\Phi_{m,n}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)} \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H'_n)},$$

where

$$\begin{aligned}\log_q X_j &\mapsto \log_q X_j, \qquad j = 1, \ldots, n; \\ \log_q X_j &\mapsto 2m - 2j + 1, \qquad j = n+1, \ldots, m.\end{aligned}$$

In particular, when $m = n$, $\Phi_{m,n}$ is the identity map.

If $n > m = r$, similarly there is a surjective homomorphism

$$\Phi'_{n,m} : \mathcal{S}(H'_n//K'_n) \to \mathcal{S}(H_m//K_m)$$

satisfying that

$$\mathcal{Z}_\sigma \circ (f' - \Phi'_{n,m}(f')) = 0$$

for all $f' \in \mathcal{S}(H'_n//K'_n)$ and $\operatorname{Re}\sigma_i \gg 0$. Using the Fourier–Satake isomorphism, the map $\Phi'_{n,m}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H'_n)} \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)},$$

where

$$\begin{aligned} \log_q X_i &\mapsto \log_q X_i, \qquad i = 1, \ldots, m; \\ \log_q X_i &\mapsto 2m - 2i + 1, \qquad i = m+1, \ldots, n. \end{aligned}$$

Lemma A.3.4. *Suppose that $\phi \in \mathcal{S}(V_m^n)$ is spherical, and $Z_\sigma(\phi) = 0$ for all $\sigma \in \mathbb{C}^r$ such that $\operatorname{Re}\sigma_i \gg 0$. Then $\omega(h')\phi$ vanishes on Σ_0 for all $h' \in H'_n$.*

Proof. It suffices to show that $\omega(h')\phi$ vanishes on $\Sigma_0^{(r)}$ since it is dense open in Σ_0. Since $\Sigma_0^{(r)}$ is transitive under the action of $\mathrm{GL}_n(E) \times H_m$, we only need to show that

$$(\omega(h', h)\phi)\left(\begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) = 0$$

for all $(h', h) \in \mathrm{GL}_n(E) \times H_m$. We can write $h' = b'k'$ with $b' \in B'_n$, $k' \in K'_n$, and $h = bk$ with $b \in B'_{m,r}$, $k \in K_m$. Since ϕ is spherical by assumption, we have

$$(\omega(h', h)\phi)\left(\begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) = (\omega(b', b)\phi)\left(\begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right) = \phi\left(\begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right)$$

with $X \in \mathrm{Mat}_{r,r}(E)$. Therefore, the lemma follows from [Ral1982, Lemma 5.2] for $k = E$. ☐

Combining Lemmas A.3.1, A.3.3 and A.3.4, we have the following proposition.

Proposition A.3.5. *The ideal*

$$\mathcal{I}_{n,m} = \{f \in \mathcal{S}(H'_n//K'_n) \otimes \mathcal{S}(H_m//K_m) \mid \omega(f) = 0\}$$

is generated by

$$\{\Phi_{m,n}(f) - f \mid f \in \mathcal{S}(H_m//K_m)\} \ \textit{(resp. } \{f' - \Phi'_{n,m}(f') \mid f' \in \mathcal{S}(H'_n//K'_n)\}\textit{)}$$

if $m \geq n$ (resp. $m < n$).

When $E = F \oplus F$, the corresponding unitary group H'_n (resp. H_m) can be identified with $\mathrm{GL}_n(F)$ (resp. $\mathrm{GL}_m(F)$). The Weil representation ω, which realizes on the space $\mathcal{S}(\mathrm{Mat}_{m,n}(F))$, is simply given by the formula $(\omega(g', g)\phi)(x) = \phi(g^{-1}xg')$ for $g' \in \mathrm{GL}_n(F)$, $g \in \mathrm{GL}_m(F)$ and $\phi \in \mathcal{S}(\mathrm{Mat}_{m,n}(F))$ (*cf.* [Ral1982, Section 6]). Without lost of generality, we assume that $n \geq m$. Then the ideal

$$\mathcal{J}_{n,m} = \{f \in \mathcal{S}(\mathrm{GL}_n(F) /\!/ \mathrm{GL}_n(\mathcal{O}_F)) \otimes \mathcal{S}(\mathrm{GL}_m(F) /\!/ \mathrm{GL}_m(\mathcal{O}_F)) \mid \omega(f) = 0\}$$

is generated by

$$\{f - \Psi_{n,m}(f) \mid f \in \mathcal{S}(\mathrm{GL}_n(F) /\!/ \mathrm{GL}_n(\mathcal{O}_F))\}.$$

In terms of the Fourier–Satake isomorphism, the surjective homomorphism $\Psi_{n,m}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(\mathrm{GL}_n(F))} \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(\mathrm{GL}_m(F))},$$

where

$$\begin{aligned} \log_q X_i &\mapsto -\log_q X_i + \frac{n-m}{2}, \qquad i = 1, \ldots, m; \\ \log_q X_i &\mapsto -i + \frac{n+1}{2}, \qquad i = m+1, \ldots, n. \end{aligned}$$

Proposition A.3.5 and its analogue in the above case imply the following corollary.

Corollary A.3.6. *Assume that $m = n$. Then the groups H'_n and H_n are isomorphic.*

1. *Let π be an unramified irreducible admissible representation of H'_n. Then the theta correspondence of π to H_n, with respect to the Weil representation ω, is nontrivial and isomorphic to π.*

2. *Let π be an unramified irreducible admissible representation of $\mathrm{GL}_n(F)$, and χ an unramified character of $F^\times$. Then the theta correspondence of π to $\mathrm{GL}_n(F)$, with respect to the Weil representation ω_χ defined by $(\omega_\chi(g', g)\phi)(x) = \chi(\det g')\phi(g^{-1}xg')$ for $\phi \in \mathcal{S}(\mathrm{Mat}_{n,n}(F))$, is nontrivial and isomorphic to $\pi^\vee \otimes \chi$.*

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